Math 454/554 Notes 9/30

We have shown that, for a continuous function f on the interval $[-c, c]$, the Fourier series represented by

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{\pi n x}{c}\right) + B_n \sin\left(\frac{\pi n x}{c}\right) \right)$$

where $A_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{\pi n x}{c}\right) dx$ and $B_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\left(\frac{\pi n x}{c}\right)\ dx$, converges to f in mean. However, recall that the heat equation

$$u_t = k u_{xx}$$

was formulated on $[0, c]$ with boundary conditions $u_x(0, t) = u_x(c, t) = 0$ and $u(x, 0) = f(x)$ for some (continuous) function f on $[0, c]$. What we'd like to show is mean convergence of the series

$$C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{\pi n x}{c}\right)$$

to f where $C_0 = \sqrt{\frac{1}{c}}\int_0^c f(x)\ dx$ and $C_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{\pi n x}{c}\right)\ dx$.

This follows by extending f to an even function on $[-c, c]$ by defining

$$g(x) = \begin{cases} f(x) & 0 \le x \le c \\ f(-x) & -c \le x < 0. \end{cases}$$

Then $g(x)$ is even and continuous on $[-c, c]$ and so the Fourier series of g converges to g in mean on that interval. However, on $[-c, c]$,

$$\int_{-c}^{c} g(x)\sin\left(\frac{\pi n x}{c}\right)\ dx = 0$$

since $g(x)\sin\left(\frac{\pi n x}{c}\right)$ is an odd function. Therefore,

$$g(x) = \frac{A_0'}{2} + \sum_{n=1}^{\infty} A_n' \cos\left(\frac{\pi n x}{c}\right)$$

in mean, where $A'_n = \frac{1}{c}\int_{-c}^{c} g(x)\cos\left(\frac{\pi n x}{c}\right)dx$. However,

$$\begin{aligned} A'_n &= \frac{1}{c}\int_{-c}^{c} g(x)\cos\left(\frac{\pi n x}{c}\right)dx \\ &= \frac{2}{c}\int_{0}^{c} g(x)\cos\left(\frac{\pi n x}{c}\right)dx \\ &= \frac{2}{c}\int_{0}^{c} f(x)\cos\left(\frac{\pi n x}{c}\right)dx \\ &= C_n, \end{aligned}$$

and so by the Parseval Formula,

$$\int_0^c (f(x))^2\,dx = 2\int_{-c}^{c}(g(x))^2\,dx = 2\sum_{n=0}^{\infty}\langle g, \sqrt{\frac{1}{c}}\cos\left(\frac{\pi n x}{c}\right)\rangle^2 = \sum_{n=0}^{\infty}\langle f, \sqrt{\frac{2}{c}}\cos\left(\frac{\pi n x}{c}\right)\rangle^2,$$

where the inner product in the first sum is over $[-c, c]$ and the one in the second over $[0, c]$. This implies that the series of all cosines converges to f in mean on the interval $[0, c]$.

A couple of definitions:

Definition: If $f \in C([0, c])$, then the *Fourier cosine series* of f is the series

$$C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{\pi n x}{c}\right)$$

where the C_n's are as above. The *Fourier sine series* of f is the series

$$\sum_{n=1}^{\infty} D_n \sin\left(\frac{\pi n x}{c}\right)$$

where $D_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{\pi n x}{c}\right)\, dx$. Both series converge in mean to f on the interval $[0, c]$.

You may object that the method of proof employed to exhibit mean convergence of the Fourier cosine series to f will fail since $f(0)$ is not necessarily zero. This does not affect the correctness of the result, though.

Recall that we arrived at the Fourier cosine series as a solution to the differential equation

$$g''(x) - \lambda g(x) = 0.$$

What is not yet clear is that if $g(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos\left(\frac{\pi n x}{c}\right)$ in mean, then the second derivative of g is equal to the term-wise second derivative of the Fourier cosine series of g.

To illustrate this point, consider the following happy example in a weaker mode of convergence:

Example: Let

$$g_n(x) = n^2 x^2 (1 - x^2)^n$$

for $x \in [0, 1]$. Then $g_n(x) = x f_n(x)$, where

$$f_n(x) = n^2 x (1 - x^2)^n.$$

In your homework, you showed $f_n \to 0$ pointwise, and so $g_n \to 0$ pointwise on $[0, 1]$. However, you can check that

$$g_n''(x) = 2n^2(1 - x^2)^n - 10x^2 n^3 (1 - x^2)^{n-1} + 4(n-1)n^3 x^4 (1 - x^2)^{n-2}$$

and so $g_n''(1) = 2n^2$, which does not converge to zero as $n \to \infty$. Then it cannot be the case that $g_n'' \to 0$ pointwise.

The previous example leads us to a discussion of *linear operators.*

Definition: A function T from a vector space V to another vector space W (both over the same field) is called *linear* if for all $v, w \in V$ and all scalars α, β,

$$T(\alpha v + \beta w) = \alpha T v + \beta T w.$$

Such a T is commonly called (by functional analysts) a *linear operator* from V to W.

Examples:

1. Any real $n \times m$ matrix from $V = \mathbb{R}^m$ to $W = \mathbb{R}^n$ is linear. The fact that the converse is true is a standard result in linear algebra.

2. If $V = W = C([a, b])$ and $g \in C([a, b])$, then the map M_g from V to itself given for $f \in C([a, b])$ by

   $$M_g(f) = fg$$

   for $f \in C([a, b])$ is linear.

3. Let V be the subspace of $C([a,b])$ consisting of all differentiable functions with continuous first derivative (we assume the existence of left-hand derivatives at $x = -c$ and right-hand derivatives at $x = c$- more on this later). Then the map D given by

$$D(f)(x) = f'(x)$$

for $f \in V$ is linear. Here, $W = C([a,b])$.

Now suppose V and W are normed linear spaces with norms $\|\cdot\|$ and $\|\cdot\|'$, respectively.

Definition: A linear map T from V to W is called *continuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ with

$$\|T(x-y)\|' < \varepsilon \text{ when } \|x-y\| < \delta.$$

Examples:

1. If $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$ are both given the Euclidean norm, then any real $n \times m$ matrix from V to W is continuous. In fact, no matter what norm you pick on finite dimensional spaces, any linear map between them will be continuous! This is badly false in the infinite dimensional case.

2. If $(V, \|\cdot\|) = (W, \|\cdot\|') = (C([a,b], \|\cdot\|_\infty)$ and M_g is as above, then M_g is continuous. It is also continuous on $(C([a,b], \|\cdot\|_2)$.

3. With either $\|\cdot\|_2$ or $\|\cdot\|_\infty$, the map D is discontinuous from the space of all differentiable functions with continuous first derivative on $[a,b]$ to $C([a,b])$.

In order to really see this last point, we'll need a reformulation of continuity for linear maps.

Theorem: A linear map T from $(V, \|\cdot\|)$ to $(W, \|\cdot\|')$ is continuous if and only if there is a real number $K > 0$ with

$$\|Tx\|' \leq K\|x\|$$

for all x in V.

It is not hard to prove either equivalence, so let's do the direction we need.
PROOF: ($\Rightarrow$) Suppose T is continuous. Note that the linearity of T implies that $T(0) = 0$. By the continuity of T at 0, there is a $\delta > 0$ so that

$$\|Tv\|' < 1 \text{ when } \|v\| < \delta.$$

If $\gamma > 0$, then for any $x \in V$, $\frac{\|x\|}{\|x\|+\gamma} < 1$, so

$$\left\| \frac{\delta x}{\|x\| + \gamma} \right\| = \delta \cdot \frac{\|x\|}{\|x\| + \gamma} < \delta$$

Then by the linearity of T,

$$\frac{\delta}{\|x\| + \gamma} \cdot \|Tx\|' = \left\| T \left(\frac{\delta x}{\|x\| + \gamma} \right) \right\|' < 1,$$

which implies that

$$\|Tx\|' < \frac{\|x\| + \gamma}{\delta}.$$

Since $\gamma > 0$ is arbitrary, we conclude that

$$\|Tx\|' \leq \frac{\|x\|}{\delta},$$

and so the result follows by setting $K = \frac{1}{\delta}$.

Now let's return to the third example. Let n be a natural number and let $f_n(x) = x^n$ on $[0, 1]$. Then $\|f_n\|_\infty = 1$ for all n, but

$$\|D(f_n)\|_\infty = \|nx^{n-1}\|_\infty = n = n\|x_n\|_\infty$$

which goes to infinity as $n \to \infty$. Therefore, D cannot be continuous as a map from a subspace of $(C([a, b], \|\cdot\|_\infty)$ to $(C([a, b], \|\cdot\|_\infty)$. Further,

$$\|f_n\|_2^2 = \int_0^1 x^{2n}\, dx = \frac{x^{2n+1}}{2n+1}\bigg|_0^1 = \frac{1}{2n+1}$$

but

$$\|D(f_n)\|_2^2 = \int_0^1 n^2 x^{2n-2}\, dx = n^2 \frac{x^{2n-1}}{2n-1}\bigg|_0^1 = \frac{n^2}{2n-1}$$

and so

$$\|D(f_n)\|_2^2 = \frac{2n-1}{2n+1} n^2 \cdot \|f_n\|_2^2$$

which again goes to infinity as $n \to \infty$. This shows that D is not continuous as a map from a subspace of $(C([a,b], \|\cdot\|_2)$ to $(C([a,b], \|\cdot\|_2)$.

However, we should like the operator D, and for the purposes of analyzing the heat equation, we should like D^2 even better. This is so because the equation

$$g''(x) - \lambda g(x) = 0.$$

can be rewritten as

$$D^2(g) - \lambda g = 0.$$

Since we are momentarily only considering convergence of Fourier series in mean, it is desirable to know that if $g_n \to g$ in mean, then $D^2(g_n) \to D^2(g)$ in mean as well. With a few assumptions, this does in fact hold, though we shall only state and not prove it.

Theorem: If $g_n \to g$ in mean on $[0, c]$ and $\{g_n\}_{n=1}^{\infty}$ are all twice-differentiable on $[0, c]$ with continuous second derivative, then if $D^2(g_n) \to h$ in mean, we have that $h = g$.

If f is twice-differentiable with continuous second derivative on $[-c, c]$, then in fact

$$\begin{aligned} f_N(x) &= \sum_{n=1}^{N} \left(A_n \left(\cos\left(\frac{\pi n x}{c}\right)\right)'' + B_n \left(\sin\left(\frac{\pi n x}{c}\right)\right)'' \right) \\ &= \sum_{n=1}^{N} \left(-A_n \frac{n^2\pi^2}{c^2} \cos\left(\frac{\pi n x}{c}\right) - B_n \frac{n^2\pi^2}{c^2} \sin\left(\frac{\pi n x}{c}\right) \right) \end{aligned}$$

is convergent in mean, and so the theorem applies to show that this sum is in fact equal to f in mean. This provides additional evidence that mean convergence is the correct mode of convergence for Fourier series of continuous functions. Further investigation of this thread leads to the theory of Sobolev spaces, which are indispensable tools in the study of partial differential equations.