The Minimum of the Dirichlet Kernel

Let

$$D_N(x) = \sum_{n=-N}^{N} e(nx) = \frac{\sin(2N+1)\pi x}{\sin \pi x} \tag{1}$$

denote the Dirichlet kernel, where $e(\theta) = e^{2\pi i\theta}$. Clearly the maximum of the Dirichlet kernel is $D_N(0) = 2N+1$. The Dirichlet kernel has zeros at $n/(2N+1)$ for $n = 1, 2, \ldots, 2N$. Between consecutive zeros, $D_N(x)$ must have at least one local extremum, making a total of at least $2N$ local extrema. But $D_n'(x)$ is a trigonometric polynomial of degree N, so it can have at most $2N$ roots. Thus we conclude that there is exactly one local extremum between each pair of consecutive zeros. The unique maximum of $D_N(x)$ in $[-1/(2N+1), 1/(2N+1)]$ is at $x = 0$. Put $x_0 = 0$, and for $n = 1, \ldots, 2N-1$, let x_n denote the unique root of $D_N'(x) = 0$ in the interval $[n/(2N+1), (n+1)/(2N+1)]$. Since $1/\sin \pi x$ is decreasing for $0 < x \le 1/2$, it follows that

$$|D_N(x_{n-1})| > |D_N(x_n - 1/2N+1)| > |D_N(x_n)|$$

for $1 \le n \le N$. The Dirichlet kernel lies between the two envelopes $\pm 1/\sin \pi x$, and is tangent to one or the other of these curves at the points $(2n-1)/(2(2N+1))$, $n = 1, 2, \ldots, 2N+1$. In particular, at the second of these points we have

$$D_N\Big(\frac{3/2}{2N+1}\Big) = \frac{-1}{\sin \frac{3\pi}{4N+2}} \sim -c_1(2N+1)$$

where

$$c_1 = \frac{2}{3\pi} = .2122065907\,.$$

One might speculate that $3/(4N+2)$ is close enough to x_1 to ensure that $D_N(x_1) \sim -c_1(2N+1)$, but we show below that the minimum is smaller:

$$D_N(x_1) \sim -c_0(2N+1) \tag{2}$$

where

$$c_0 = .2172336282\,. \tag{3}$$

Clearly

$$D_N'(x) = \pi\frac{(2N+1)\sin\pi x\,\cos\pi(2N+1)x - \cos\pi x\,\sin\pi(2N+1)x}{(\sin\pi x)^2}\,. \tag{4}$$

For $0 \le x \le 1$ we have

$$\sin\pi x = \pi x + O\big(x^3\big), \qquad \cos\pi x = 1 + O(x^2)\,.$$

Hence the numerator in (4) is

$$\pi(2N+1)x\cos\pi(2N+1)x - \sin\pi(2N+1)x + O(Nx^3) + O(x^2)\,. \tag{5}$$

Put $u = \pi(2N+1)x$. The main term above is $f(u) = u\cos u - \sin u$. We note that $f(\pi) = -\pi$, that $f(3\pi/2) = \pi$, and that $f'(u) = -u\sin u \ge 0$ for $\pi \le u \le 3\pi/2$. Hence f has precisely one zero in the interval $(\pi, 3\pi/2)$, say $f(u') = 0$. We apply Newton's method, which is to say we start with $u_1 = 4.6$, and set

$$u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)} = u_n + \cot u - \frac{1}{u}\,.$$

This generates the following numbers:

n	u_n	$u_n - u'$
1	4.6	10^{-1}
2	4.495473284870517486798728632548	2×10^{-3}
3	4.493410402318926518824251478832	10^{-6}
4	4.493409457909262667896829189473	2×10^{-13}
5	4.493409457909064175307880936048	10^{-26}

Put

$$x' = \frac{u'}{\pi(2N+1)}\,.$$

If we take $x = x'$ in (5), then the main terms sum to 0, and the error terms are $O(N^{-2})$. The denominator in (4) is also $\asymp N^{-2}$, so $D_N'(x') = O(1)$. Since $D_N''(x) \asymp N^3$ for x near x', it follows that $x_1 = x' + O(N^{-3})$, and hence that $D_N(x_1) = D_N(x') + O(N^{-3})$. From the equation $u' = \tan u'$ we deduce that $\sin u' = -u'/\sqrt{1+u'^2}$. Since

$$\sin\pi(2N+1)x_1 = \sin\pi(2N+1)x' + O(N^{-2}) = \sin u' + O(N^{-2}) = \frac{-u'}{\sqrt{1+u'^2}} + O(N^{-2}),$$

and since

$$\sin\pi x_1 \sim \sin\pi x' = \sin\frac{u'}{2N+1} \sim \frac{u'}{2N+1},$$

it follows that we have (2) with

$$c_0 = \frac{1}{\sqrt{1+u'^2}} = 0.21723362821122165741\,.$$