

Derivation of the heat equation

We will consider a rod so thin that we can effectively think of it as one-dimensional and lay it along the x axis, that is, we let the coordinate x denote the position of a point in the rod. We will imagine that the temperature at every point along the rod is known at some initial time $t = 0$ and we will be interested in determining the temperature at future times. The rod is assumed to be wrapped in some sort of insulation so that heat can't escape but must flow along the x axis. The rod is also assumed to be "homogeneous" in the sense that its physical composition is the same at all points. We will denote the temperature at position x and time t by $u(x, t)$.

Now, heat and temperature are not the same thing. Temperature is a property describing the relative hotness or coldness of an object that can be measured (in degrees on some chosen scale) by comparison with another object (*e.g.*, a thermometer). Heat is energy. It is usually measured in calories or BTU (British Thermal Units). The connection is simple: to raise or lower the temperature of something requires a gain or loss of energy. How much energy is needed depends on a property of the object called the *specific heat*.

Associated with our homogeneous rod is a constant σ (the specific heat) that depends on the substance the rod is made of and gives a measure of the amount of heat needed to raise the temperature of one unit mass by one degree. Since it is more convenient to deal with lengths of the rod rather than masses, we will introduce another constant δ , the mass density of the rod. It represents the amount of mass in one unit length of the rod. So, if the rod has length L , the total mass of the rod is δL . If we are to change the temperature of a rod of length L by T degrees, the energy needed (or lost) is $\sigma T \delta L$.

Consider a point x in the rod and the small section of length Δx of the rod between x and $x + \Delta x$. Later, we will take a limit as $\Delta x \rightarrow 0$, so we are free to think of Δx as being as small as we like. Let's begin by thinking of it small enough so that the temperature of the section is pretty well approximated by the temperature of its midpoint, that is, the temperature at time t of the section is just $u(x + \frac{\Delta x}{2}, t)$. The temperature at a later time $t + \Delta t$ is then just $u(x + \frac{\Delta x}{2}, t + \Delta t)$, so the temperature difference at these two times is

$$u(x + \frac{\Delta x}{2}, t + \Delta t) - u(x + \frac{\Delta x}{2}, t).$$

Now, how much energy (heat) does it take to change the temperature of the small section of the rod by this amount? Well, according to the definition of specific heat, we can write this energy as:

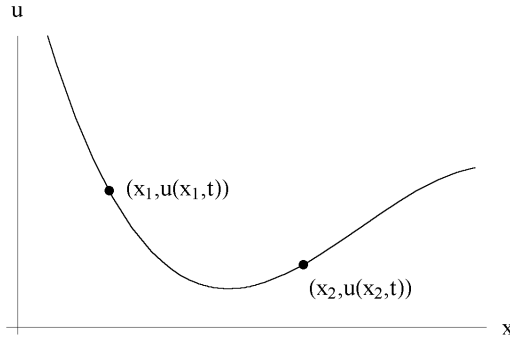
$$\sigma \left[u(x + \frac{\Delta x}{2}, t + \Delta t) - u(x + \frac{\Delta x}{2}, t) \right] \delta \Delta x. \quad (1)$$

There is another way to arrive at an equivalent expression for the amount of heat needed given above by thinking in terms of heat conduction. Substances conduct heat differently, this is why some substances are better insulators than others. Experiment shows that the amount of heat conducted through a point x is proportional to the difference of temperatures on either side of x . The constant of proportionality is called the thermal conductivity of the substance and will be denoted by k . Physicists call this conduction "heat flux" and it is expressed mathematically as follows: the amount of heat per unit time crossing the point x in the positive x direction is given by

$$\Phi(x, t) = -k \frac{\partial u}{\partial x}(x, t). \quad (2)$$

To understand the definition of heat flux it may be useful to consider a picture. In the figure below we graph the temperature $u(x, t)$ as a function of x for some fixed value of t . The partial derivative $\frac{\partial u}{\partial x}(x, t)$ is then just the regular derivative with respect to x , that is, it represents the slope of the tangent line to the graph of u at the point $(x, u(x, t))$. At the point labeled $(x_1, u(x_1, t))$, this slope is negative and equation (2) tells us then that a positive amount of heat per unit time will flow past x_1 in the positive x direction. This makes sense, as it is hotter just to the left of x_1 than it is just to the right. At the point labeled $(x_2, u(x_2, t))$, the slope is positive and equation (2) tells us that a negative amount of heat per unit time will flow past

x_2 in the positive x direction, in other words, heat is flowing past x_2 in the negative direction. This makes sense, as it is colder just to the left of x_1 than it is just to the right.



Now back to our small subsection of the rod between x and $x + \Delta x$. The total amount of heat flowing into the section of the rod per unit time is just the heat flux at x minus the heat flux at $x + \Delta x$ (the minus here is because of the direction: at $x + \Delta x$, “into the section” is in the negative x direction), or, $\Phi(x, t) - \Phi(x + \Delta x, t)$. Over a small time interval between t and $t + \Delta t$, the total energy coming into the small section of the rod is then

$$[\Phi(x, t) - \Phi(x + \Delta x, t)] \Delta t = k \left[\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right] \Delta t. \quad (3)$$

Equating (1) and the right side of (3) gives

$$\sigma \left[u\left(x + \frac{\Delta x}{2}, t + \Delta t\right) - u\left(x + \frac{\Delta x}{2}, t\right) \right] \delta \Delta x = k \left[\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right] \Delta t$$

or,

$$\frac{u\left(x + \frac{\Delta x}{2}, t + \Delta t\right) - u\left(x + \frac{\Delta x}{2}, t\right)}{\Delta t} = \kappa \frac{\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)}{\Delta x} \quad (4)$$

where $\kappa \equiv \frac{k}{\sigma \delta}$. If we now take the limit of both sides of (4) as $\Delta x \rightarrow 0$, we obtain:

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \kappa \frac{\partial^2 u}{\partial x^2}(x, t).$$

Taking the limit of this last equation as $\Delta t \rightarrow 0$ now gives us what is commonly known as the one-dimensional heat (or diffusion) equation:

$$\frac{\partial u}{\partial t}(x, t) = \kappa \frac{\partial^2 u}{\partial x^2}(x, t). \quad (5)$$