Lecture 3: Convergence of Fourier Series*

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Let $f$ be an absolutely integrable function on $\mathbb{T} := [-\pi, \pi]$, i.e., $f \in L^1(\mathbb{T})$. For $n = 1, 2, \ldots$ define

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) e^{-in\theta} d\theta.$$ 

The series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$ is called the Fourier Series (FS) of $f$.

The FS of $f$ does not converge to $f$ point-wise a.e.. If we restrict to continuous functions, the point-wise convergence of FS can not be guaranteed. In this lecture, we define weaker but important criteria for convergence under which the FS of continuous $f$ does converge to $f$.

A. The definition of convolution given in Lecture 1 can be easily modified for functions in $L^1(\mathbb{T})$. For $f, g \in L^1(\mathbb{T})$ the convolution $f \ast g$ is defined as

$$f \ast g(x) = \int_{\mathbb{T}} f(x-y)g(y) dy,$$

where $f(x) = f(x \mod \mathbb{T})$. Also, Theorem 1 of Lecture 1 and property (v) of convolution hold. Therefore, we can mimic the proof therein to show there is no identity for convolution in $L^1(\mathbb{T})$, provided we prove the Riemann-Lebesgue Lemma. Here we give an alternate proof of Riemann-Lebesgue Lemma using approximate identity.

**Definition 1.** A sequence $\{K_n\}$ of continuous functions on $\mathbb{R}$ is a summability kernel if for all $n \in \mathbb{N}$

(i) $\int_{\mathbb{R}} K_n(x) \, dx = 1$,

(ii) $\int_{\mathbb{R}} |K_n(x)| \, dx \leq M$,

(iii) For each $0 < \delta < \pi$, $\lim_{n \to \infty} \int_{|x| > \delta} |K_n(x)| \, dx = 0$.

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Note here that limits of integral in property three represents the union of regions \([-\pi, -\delta]\) and \((\delta, \pi]\). As was shown in the last lecture, a summability kernel is a bounded approximate identity for \(L^1(\mathbb{T})\). Also, the Fejer kernel
\[
K_n(t) = \sum_{j=-n}^{n} \frac{1}{2\pi} \left(1 - \frac{|j|}{n+1}\right) e^{ijt} = \frac{1}{2(n+1)\pi} \left(\frac{\sin \frac{n+1}{2}t}{\sin \frac{1}{2}t}\right)^2
\]
is a summability kernel. Therefore we have the following lemma.

**Lemma 1.** For \(f \in L^1(\mathbb{T})\) and \(n = 1, 2, \ldots\), \(\sigma_n(f) := K_n \ast f\). Then \(\sigma_n(f) \to f\) in \(L^1(\mathbb{T})\).

Note that \(\sigma_n(f)\) is a function of form \(\sum_{n=-N}^{N} a_n e^{int}\). Such a function is called a *trigonometric polynomial* of degree \(N\). The Uniqueness Theorem for FS follows from this Lemma.

The Lemma above says that for all functions \(f\) in \(L^1(\mathbb{T})\) and all positive \(\epsilon\), we can find a trigonometric polynomial \(P\) such that \(\|f - P\|_{L^1} < \epsilon\).

It can be simply verified that for trigonometric polynomial \(P\) of degree \(N\), \(\hat{P}(n) = 0\) for all \(n\) greater than \(N\). The Riemann-Lebesgue Lemma now follows.

**Theorem 2.** \(^1\) Let \(f \in L^1(\mathbb{T})\), then
\[
\lim_{|n| \to \infty} \hat{f}(n) = 0.
\]

**Proof:** Let \(\epsilon > 0\) and let \(P\) be a trigonometric polynomial on \(\mathbb{T}\) such that \(\|f - P\|_{L^1} < \epsilon\). If \(|n| > \text{degree of } P\), then
\[
|\hat{f}(n)| = |(\hat{f} - \hat{P})(n)| \leq \|f - P\|_{L^1} < \epsilon.
\]

\[\square\]

**B.** Now we restrict to continuous functions \(f\) on \(\mathbb{T}\). Since \(\mathbb{T}\) is compact, \(f\) is bounded. Therefore we get following strengthening of Theorem 5 of Lecture 1.

**Theorem 3.** For a summability kernel \(\{K_n\}_{n \in \mathbb{N}}\) and a continuous function \(f\) on \(\mathbb{T}\), \(K_n \ast f\) converges to \(f\) point-wise.

Proof: Let \( \epsilon > 0 \). Choose \( \delta > 0 \) such that if \(|t| \leq \delta\), then \( \|f - f(t)\|_{L^\infty} < (\epsilon/2M) \). This can be done since \( f \) is continuous and \( \mathbb{T} \) is compact imply \( f \) is uniformly continuous. Then for \( t \) in \( \mathbb{T} \) we have

\[
|K_n * f(t) - f(t)| \leq \int_{\mathbb{R}} |K_n(y)| \times |f(t - y) - f(t)| \, dy \quad \text{[Using} \int_{\mathbb{R}} K_n(x) \, dx = 1]\]

\[
= \int_{|y| \leq \delta} |K_n(y)| \times |f(t - y) - f(t)| \, dy + \int_{|y| > \delta} |K_n(y)| \times |f(t - y) - f(t)| \, dy \leq \frac{\epsilon}{2} + 2\|f\|_{L^\infty} \int_{|y| > \delta} |K_n(y)| \, dy \leq \epsilon,
\]

for sufficiently large \( n \).

\[\square\]

C. The Theorem will be instrumental in the proof of convergence of FS. We begin by defining the convergence criteria we shall be interested in.

Definition 2. A series \( \sum_{n} x_n \) with complex terms \( x_n \) for \( n = 1, 2, \ldots \), is said to be Abel summable to \( L \) if the following holds:

(i) For every \( 0 < r < 1 \), the series \( \sum_{n=1}^{\infty} r^n x_n \) converges to a limit \( L(r) \).

(ii) \( L(r) \) converges to \( L \) as \( r \to 1 \).

Definition 3. For a series \( \sum_{n} x_n \) with complex terms \( x_n \) for \( n = 1, 2, \ldots \), define \( s_N = \sum_{n=1}^{N} x_n \) and \( \sigma_n = \frac{s_1 + \ldots + s_n}{n} \). Then the series \( \sum_{n} x_n \) is said to be Cesáro summable to \( L \) [also called \((C,1)\)summable to \( L \)] if the sequence \( \sigma_n \) converges to \( L \) as \( n \to \infty \).

We state the next result without proof.

Theorem 4. A series \( \sum_{n} x_n \) with complex terms \( x_n \) for \( n = 1, 2, \ldots \) converges to \( L \) implies it is Cesáro summable to \( L \). Furthermore, \( \sum_{n} x_n \) is Cesáro summable to \( L \) implies it is Abel summable to \( L \).

Consider the FS of a continuous function \( f \) on \( \mathbb{T} \). Define the partial sums

\[
s_N(f; \theta) = \sum_{n=-N}^{N} \hat{f}(n)e^{i\theta n},
\]
\[ N = 0, 1, 2, \ldots \] Also, for \( e_n(t) = e^{int} \), \( f * e_n(\theta) = 2\pi \hat{f}(n) e^{in\theta} \). Therefore, \( s_N(f; \theta) = f * D_N(\theta) \) where

\[
D_N(\theta) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{int}.
\]

\( D_N(t) \) is called the Dirichlet kernel. If the Dirichlet kernel was a summability kernel we could have shown the point-wise convergence of FS of \( f \) to \( f \). However this is not true. To show that FS of \( f \) is Cesaro summable to \( f \), we need to show that \( \frac{1}{N+1} \sum_{n=0}^{N} s_n(f; \theta) \) converges to \( f(\theta) \). Note that

\[
\frac{1}{N+1} \sum_{n=0}^{N} s_n(f; \theta) = f * \frac{1}{N+1} \sum_{n=0}^{N} D_n(\theta).
\]

Next we show that \( \frac{1}{N+1} \sum_{n=0}^{N} D_n \) is the Fejer kernel! First note that

\[
\frac{1}{N+1} \sum_{n=0}^{N} D_n(t) = \frac{1}{2(N+1)\pi} \sum_{n=0}^{N} \sum_{k=-n}^{n} e^{ikt} = \frac{1}{2(N+1)\pi} \sum_{n=0}^{N} [1 + 2 \sum_{k=1}^{n} \cos kt].
\]

Multiplying and dividing \( [1 + 2 \sum_{k=1}^{n} \cos kt] \) with \( \sin t/2 \) we get

\[
1 + 2 \sum_{k=1}^{n} \cos kt = \frac{1}{\sin t/2} [\sin t/2 + 2 \sum_{k=1}^{n} \cos kt \sin t/2] = \frac{\sin(n + 1/2)t}{\sin t/2}.
\]

Therefore,

\[
\frac{1}{N+1} \sum_{n=0}^{N} D_n(t) = \frac{1}{2(N+1)\pi} \sum_{n=0}^{N} \frac{\sin(n + 1/2)t}{\sin t/2}
\]
Again, multiplying and dividing by \( \sin t/2 \) we get

\[
\frac{1}{N+1} \sum_{n=0}^{N} D_n(t) = \frac{1}{2(N+1)\pi} \sum_{n=0}^{N} \frac{\sin(n + 1/2)t \sin t/2}{(\sin t/2)^2}
\]

\[
= \frac{1}{2(N+1)\pi} \sum_{n=0}^{N} \frac{\sin(n + 1/2)t \sin t/2}{(\sin t/2)^2}
\]

\[
= \frac{1}{2(N+1)\pi} \left( \frac{\sin \left(\frac{(N+1)t}{2}\right)}{\sin \left(\frac{t}{2}\right)} \right)^2.
\]

Since Fejer kernel is a summability kernel, we have the following result.

**Theorem 5.** The FS of \( f \) is Cesàro summable to \( f \).

Note that this, using Theorem 4 also implies that FS of \( f \) is Abel summable to \( f \). This result can also be obtained directly. The proof entails analysis of what is known as the Poisson kernel. [You might have encountered this already in partial differential equations]