Lecture 3: Convergence of Fourier Series*

Himanshu Tyagi

Let f be an absolutely integrable function on $\mathbb{T} := [-\pi, \pi]$, i.e., $f \in L^1(\mathbb{T})$. For $n = 1, 2, \ldots$ define

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) e^{-in\theta}\, d\theta.$$

The series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$ is called the *Fourier Series* (FS) of f.

The FS of f does not converge to f point-wise a.e.. If we restrict to continuous functions, the point-wise convergence of FS can not be guaranteed. In this lecture, we define weaker but important criteria for convergence under which the FS of continuous f does converge to f.

A. The definition of convolution given in Lecture 1 can be easily modified for functions in $L^1(\mathbb{T})$. For $f, g \in L^1(\mathbb{T})$ the *convolution* $f * g$ is defined as

$$f * g(x) = \int_{-\pi}^{\pi} f(x - y) g(y)\, dy,$$

where $f(x) = f(x \mod \mathbb{T})$. Also, Theorem 1 of Lecture 1 and property (v) of convolution hold. Therefore, we can mimic the proof therein to show there is no identity for convolution in $L^1(\mathbb{T})$, provided we prove the Riemann-Lebesgue Lemma. Here we give an alternate proof of Riemann-Lebesgue Lemma using approximate identity.

Definition 1. A sequence $\{K_n\}$ of continuous functions on $\mathbb{R}$ is a *summability kernel* if for all $n \in \mathbb{N}$

(i) $\int_{\mathbb{R}} K_n(x)\, dx = 1$,

(ii) $\int_{\mathbb{R}} |K_n(x)|\, dx \leq M$,

(iii) For each $0 < \delta < \pi$, $\lim_{n \to \infty} \int_{|x| > \delta} |K_n(x)|\, dx = 0$.

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Note here that limits of integral in property three represents the union of regions $[-\pi, -\delta)$ and $(\delta, \pi]$. As was shown in the last lecture, a summability kernel is a bounded approximate identity for $L^1(\mathbb{T})$. Also, the Fejer kernel $K_n(t) = \sum_{j=-n}^{n} \frac{1}{2\pi}\left(1 - \frac{|j|}{n+1}\right) e^{ijt} = \frac{1}{2(n+1)\pi}\left(\frac{\sin\frac{n+1}{2}t}{\sin\frac{1}{2}t}\right)^2$ is a summability kernel. Therefore we have the following lemma.

Lemma 1. *For $f \in L^1(\mathbb{T})$ and $n = 1, 2, \ldots$, $\sigma_n(f) := K_n * f$. Then $\sigma_n(f) \to f$ in $L^1(\mathbb{T})$.*

Note that $\sigma_n(f)$ is a function of form $\sum_{n=-N}^{N} a_n e^{int}$. Such a function is called a *trigonometric polynomial* of *degree* N. The Uniqueness Theorem for FS follows from this Lemma.
The Lemma above says that for all functions f in $L^1(\mathbb{T})$ and all positive ϵ, we can find a trigonometric polynomial P such that $\|f - P\|_{L^1} < \epsilon$.
It can be simply verified that for trigonometric polynomial P of degree N, $\hat{P}(n) = 0$ for all n greater than N. The Riemann-Lebesgue Lemma now follows.

Theorem 2. [1] *Let $f \in L^1(\mathbb{T})$, then*

$$\lim_{|n|\to\infty} \hat{f}(n) = 0.$$

Proof: Let $\epsilon > 0$ and let P be a trigonometric polynomial on $\mathbb{T}$ such that $\|f - P\|_{L^1} < \epsilon$. If $|n| >$ degree of P, then

$$|\hat{f}(n)| = |(f \hat{-} P)(n)| \leq \|f - P\|_{L^1} < \epsilon.$$

□

B. Now we restrict to continuous functions f on $\mathbb{T}$. Since $\mathbb{T}$ is compact, f is bounded. Therefore we get following strengthening of Theorem 5 of Lecture 1.

Theorem 3. *For a summability kernel $\{K_n\}_{n\in\mathbb{N}}$ and a continuous function f on $\mathbb{T}$, $K_n * f$ converges to f point-wise.*

[1] This proof is taken from Y. Katznelson, *Introduction to harmonic analysis*, third edition, Cambridge University Press, 2004.

Proof: Let $\epsilon > 0$. Choose $\delta > 0$ such that if $|t| \leq \delta$, then $\|f - f^{(t)}\|_{L^\infty} < (\epsilon/2M)$. This can be done since f is continuous and $\mathbb{T}$ is compact imply f is uniformly continuous. Then for t in $\mathbb{T}$ we have

$$\begin{aligned}
|K_n * f(t) - f(t)| &\leq \int_{\mathbb{R}} |K_n(y)| \times |f(t-y) - f(t)|\, dy \qquad [\text{Using } \int_{\mathbb{R}} K_n(x)\, dx = 1] \\
&= \int_{|y|\leq\delta} |K_n(y)| \times |f(t-y) - f(t)|\, dy + \int_{|y|>\delta} |K_n(y)| \times |f(t-y) - f(t)|\, dy \\
&\leq \frac{\epsilon}{2} + 2\|f\|_{L^\infty} \int_{|y|>\delta} |K_n(y)|\, dy \\
&\leq \epsilon,
\end{aligned}$$

for sufficiently large n.

$\square$

C. The Theorem will be instrumental in the proof of convergence of FS. We begin by defining the convergence criteria we shall be interested in.

Definition 2. A series $\sum_n x_n$ with complex terms x_n for $n = 1, 2, \ldots$, is said to be *Abel summable* to L if the following holds:

(i) For every $0 < r < 1$, the series $\sum_{n=1}^{\infty} r^n x_n$ converges to a limit $L(r)$.

(ii) $L(r)$ converges to L as $r \to 1$.

Definition 3. For a series $\sum_n x_n$ with complex terms x_n for $n = 1, 2, \ldots$, define $s_N = \sum_{n=1}^{N} x_n$ and $\sigma_n = \frac{s_1 + \ldots + s_n}{n}$. Then the series $\sum_n x_n$ is said to be *Cesáro summable* to L [also called $(C, 1)$-summable to L] if the sequence σ_n converges to L as $n \to \infty$.

We state the next result without proof.

Theorem 4. *A series $\sum_n x_n$ with complex terms x_n for $n = 1, 2, \ldots$ converges to L implies it is Cesáro summable to L.*
Furthermore, $\sum_n x_n$ is Cesáro summable to L implies it is Abel summable to L.

Consider the FS of a continuous function f on $\mathbb{T}$. Define the partial sums

$$s_N(f; \theta) = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta},$$

$N = 0, 1, 2, \ldots$. Also, for $e_n(t) = e^{int}$, $f * e_n(\theta) = 2\pi \hat{f}(n) e^{in\theta}$. Therefore, $s_N(f;\theta) = f * D_N(\theta)$ where

$$D_N(\theta) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{int}.$$

$D_N(t)$ is called the *Dirichlet kernel*. If the Dirichlet kernel was a summability kernel we could have shown the point-wise convergence of FS of f to f. However this is not true. To show that FS of f is Cesáro summable to f, we need to show that $\frac{1}{N+1} \sum_{n=0}^{N} s_n(f;\theta)$ converges to $f(\theta)$. Note that

$$\frac{1}{N+1} \sum_{n=0}^{N} s_n(f;\theta) = f * \frac{1}{N+1} \sum_{n=0}^{N} D_n(\theta).$$

Next we show that $\frac{1}{N+1} \sum_{n=0}^{N} D_n$ is the Fejer kernel! First note that

$$\begin{aligned}
\frac{1}{N+1} \sum_{n=0}^{N} D_n(t) &= \frac{1}{2(N+1)\pi} \sum_{n=0}^{N} \sum_{k=-n}^{n} e^{ikt} \\
&= \frac{1}{2(N+1)\pi} \sum_{n=0}^{N} [1 + 2 \sum_{k=1}^{n} \cos kt].
\end{aligned}$$

Multiplying and dividing $[1 + 2 \sum_{k=1}^{n} \cos kt]$ with $\sin t/2$ we get

$$\begin{aligned}
1 + 2 \sum_{k=1}^{n} \cos kt &= \frac{1}{\sin t/2} [\sin t/2 + 2 \sum_{k=1}^{n} \cos kt \sin t/2] \\
&= \frac{\sin(n + 1/2)t}{\sin t/2}.
\end{aligned}$$

Therefore,

$$\frac{1}{N+1} \sum_{n=0}^{N} D_n(t) = \frac{1}{2(N+1)\pi} \sum_{n=0}^{N} \frac{\sin(n + 1/2)t}{\sin t/2}$$

Again, multiplying and dividing by $\sin t/2$ we get

$$\begin{aligned}\frac{1}{N+1}\sum_{n=0}^{N} D_n(t) &= \frac{1}{2(N+1)\pi}\sum_{n=0}^{N}\frac{\sin(n+1/2)t\sin t/2}{(\sin t/2)^2}\\ &= \frac{1}{2(N+1)\pi}\sum_{n=0}^{N}\frac{\sin(n+1/2)t\sin t/2}{(\sin t/2)^2}\\ &= \frac{1}{2(N+1)\pi}\left(\frac{\sin\frac{(N+1)t}{2}}{\sin\frac{t}{2}}\right)^2.\end{aligned}$$

Since Fejer kernel is a summability kernel, we have the following result.

Theorem 5. *The FS of f is Cesáro summable to f.*

Note that this, using Theorem 4 also implies that FS of f is Abel summable to f. This result can also be obtained directly. The proof entails analysis of what is known as the *Poisson kernel*. [You might have encountered this already in partial differential equations]