Convergence of Fourier series in the mean of L_2 and L_1

Gram-Schmidt orthonormalization: Suppose $\{\psi_k : k \in \mathbf{Z}_+\}$ is linearly independent. $\varphi_1 := \psi_1 / |\psi_1|_2$, $\varphi_2 := \psi_2 - \langle \psi_2, \varphi_1 \rangle$, normalize φ_2, $\varphi_3 := \psi_3 - \langle \psi_3, \varphi_2 \rangle - \langle \psi_3, \varphi_1 \rangle$, normalize φ_3, etc. Then $\{\varphi_k : k \in \mathbf{Z}_+\}$ is orthonormal.
This shows that starting with a Riesz basis we can produce an orthonormal one.
From now on $\{\varphi_k : k \in \mathbf{Z}_+\}$ will be considered orthonormal.

Approximation: $\varepsilon := \left| u - \sum_{k=1}^{n} c_k \varphi_k \right|_2$ is minimized by $c_k = \widehat{u}_k := \langle u, \varphi_k \rangle$. Proof: $\langle \varepsilon, \varepsilon \rangle = |u|_2^2 - \sum_{k=1}^{n} |\widehat{u}_k|^2 + \sum_{k=1}^{n} |\widehat{u}_k - c_k|^2$.

Bessel's inequality: $\left| u - \sum_{k=1}^{n} \widehat{u}_k \varphi_k \right|_2^2 = |u|_2^2 - \sum_{k=1}^{n} |\widehat{u}_k|^2$, so $\sum_{k=1}^{n} |\widehat{u}_k|^2 \le |u|_2^2$.

Convergence: $\sum_{k=1}^{\infty} \widehat{u}_k \varphi_k \to u$ in the mean.

Proof: $\left| \sum_{k=m}^{n} c_k \varphi_k \right|_2^2 = \sum_{k=m}^{n} |c_k|^2$, so by Bessel's inequality and the Riesz-Fischer theorem the the series converges in the mean.
Let v denote the limit. Then $\langle v, \varphi_k \rangle = \langle u, \varphi_k \rangle$, so $u - v \perp$ to all φ_k.

Parseval's formula: [1] $\langle \widehat{u}, \widehat{v} \rangle = \langle u, v \rangle$. The Fourier transform is a linear isometry.
Proof: As $n \to \infty$, Bessel's inequality becomes an equality, i.e. $|\widehat{u}|_2 = |u|_2$. Now use the polarization formula.

Hausdorff-Young inequality: $|\widehat{u}|_q \le |u|_p$ for $1 \le p \le 2$.

Fourier basis: $\{\exp(ik\theta) : k \in \mathbf{Z}\}$ is an orthonormal Riesz basis for $L_2(\mathbf{T})$. Proof: exercise.

Fourier transform: $\chi := 1_{\mathbf{T}}$, $\varphi_k := \chi^k(\exp(i\theta)) = \exp(ik\theta)$

- $\mathscr{F} : L_2(\mathbf{T}) \to \ell_2$, $\mathscr{F} u := \widehat{u}$, where $\widehat{u}_k := \langle u, \varphi_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\theta) \exp(-ik\theta)\, d\theta$. Inversion: $u(\theta) = \sum_{k=-\infty}^{\infty} \widehat{u}_k \exp(ik\theta)$
- $\mathscr{F} : L_2(\mathbf{R}) \to L_2(\mathbf{R})$, $\mathscr{F} u := \widehat{u}$, where $\widehat{u}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x) \exp(-i\omega x)\, dx$. Inversion: $u(x) = \int_{-\infty}^{\infty} \widehat{u}(\omega) \exp(i\omega x)\, dx$

Properties of the Fourier transform on $L_1(\mathbf{T})$: (since $|\exp(ik\theta)| = 1$, Fourier transform makes sense on $L_1(\mathbf{T})$)

- $\mathscr{F}$ is linear.
- $\widehat{\overline{u}}(k) = \overline{\widehat{u}(-k)}$
- Shift: $u_\tau(\theta) := u(\theta - \tau)$. $\widehat{u_\tau}(k) = \widehat{u}(k) \exp(-ik\tau)$.
- Convolution: $u * v := \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\theta - \tau) v(\tau)\, d\tau$. [2] $\widehat{u * v} = \widehat{u}\widehat{v}$.
- $\varphi_k * u = \widehat{u}_k \varphi_k$

Linear functionals: A linear functional is a linear map on a functional space to $\mathbf{C}$.
A linear functional A is called bounded (continuous) if $\exists M$ with $|Au| \le M\,|u|$.
Representable functionals: given $v \in L_2$ the map $L_2 \to \mathbf{C}$ defined by $u \mapsto \langle u, v \rangle$ is a bounded linear functional.

Riesz representation theorem: [3] Every bounded linear functional on L_2 is representable (uniquely).

The linear functional $u \mapsto u(0)$ is not representable. It is called the Dirac δ-functional and denoted by δ.
By abuse of notation it is sometimes called a function and some write $\int u(x)\delta(x)\, dx = u(0)$. Fact: $\delta * u = u$.

Summability kernels: A summability kernel on $L_1(\mathbf{T})$ is a uniformly bounded sequence of $K_n \in L_1$ with integrals 1 such that $K_n \to \delta$ (the Dirac comb), i.e. $\langle \varphi, K_n \rangle \to \varphi(0)$. In this case $K_n * u \to u$.

Fejér kernel and Cesaro sums: $K_n := \sum_{k=-n}^{n} \left[1 - \frac{|k|}{n+1} \right] \exp(ik\theta) = \frac{1}{n+1} \left[\frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{1}{2}\theta\right)} \right]^2$

Proof: $\sin^2\left(\frac{1}{2}\theta\right) = \frac{1}{2}(1 - \cos\theta) = -\frac{1}{4}\exp(-i\theta) + \frac{1}{2} - \frac{1}{4}\exp(i\theta)$. Now multiply.
$\sigma_n := K_n * u = \frac{1}{n+1} \sum_{k=0}^{n} s_k$, where $s_k := \sum_{m=-k}^{k} \widehat{u}_j \exp(im\theta)$ are the partial Fourier sums.
Cesaro sums converge to u in L_1 norm, so trigonometric polynomials are dense in $L_1(\mathbf{T})$. Also $\widehat{u} = 0 \Rightarrow u = 0$.

References:

F. Riesz, B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar, 1955 (Dover, 1990)
Y. Katznelson, *An Introduction to Harmonic Analysis*, Wiley, 1968 (Dover, 1976)

[1] For the continuous Fourier transform this is known as Plancherel's theorem
[2] Convolution is associative, commutative, and distributes over +.
[3] due to Fréchet and Riesz (1907)