**Math 331 Assignment 4**

**Due Tuesday, November 20**

In this homework set, you are free to use all the primitives, axioms, definitions, and results proved in Chapters 2-6.

**1)** (#28, Chapter 4 Exercises) Recall that a quadrilateral $\square ABCD$ is formed from four distinct points (called the *vertices*), no three of which are collinear, and from the segments $AB$, $CB$, $CD$, and $DA$ (called the *sides*), which have no intersections except at those endpoints labeled by the same letter. The notation for this quadrilateral is not unique- e.g., $\square ABCD$=$\square CBAD$. Two vertices that are endpoints of a side are called *adjacent*; otherwise the two vertices are called *opposite*. The remaining pair of segments $AC$ and $BD$ formed from the four points are called *diagonals* of the quadrilateral; they may or may not intersect at some fifth point. If $X$, $Y$, $Z$ are the vertices of $\square ABCD$ such that $Y$ is adjacent to both $X$ and $Z$, then $\sphericalangle XYZ$ is called an *angle* of the quadrilateral; if $W$ is the fourth vertex, then $\sphericalangle XWZ$ and $\sphericalangle XYZ$ are called *opposite* angles.

The quadrilaterals of main interest are the *convex* ones. By definition, they are the quadrilaterals such that each pair of opposite sides, e.g., $AB$ and $CD$, has the property that $CD$ is contained in one of the half-planes bounded by the line through $A$ and $B$, and $AB$ is contained in one of the half-planes bounded by the line through $C$ and $D$.

a) Using Pasch's theorem, prove that if one pair of opposite sides has this property, then so does the other pair of opposite sides.

b) Prove, using the crossbar theorem, that the following are equivalent:

i) The quadrilateral is convex.

ii) Each vertex of the quadrilateral lies in the interior of the opposite angle.

iii) The diagonals of the quadrilateral meet.

c) Prove that Saccheri and Lambert quadrilaterals are convex.

d) Draw a non-convex quadrilateral.

**2)** (#32, Chapter 4 Exercises) In Figure 4.33, the angle pairs ($\sphericalangle A'B'B''$, $\sphericalangle ABB''$) and ($\sphericalangle C'B'B'', \sphericalangle CBB''$) are called pairs of *corresponding angles* cut off on $l$ and $l'$ by transversal $t$. Prove that corresponding angles are congruent if and only if alternate interior angles are congruent.

**3)** (#10, Chapter 5 Exercises) *Fundamental theorem on similar triangles.* Given $\triangle ABC \sim \triangle A'B'C'$; i.e., given $\sphericalangle A \cong \sphericalangle A'$, $\sphericalangle B \cong \sphericalangle B'$, and $\sphericalangle C \cong \sphericalangle C'$. Then corresponding sides are proportional; i.e., $\overline{AB}/\overline{A'B'} = \overline{AC}/\overline{A'C'} = \overline{BC}/\overline{B'C'}$ (see figure 5.14). Prove the theorem. (Hint: Let $B''$ be the point on $\overrightarrow{AB}$ such that $AB'' \cong A'B'$ and let $C''$ be the point on $\overrightarrow{AC}$ such that $AC'' \cong A'C'$. Use the hypothesis to show that $\triangle AB''C'' \cong \triangle A'B'C'$ and deduce from corresponding angles that $\overleftrightarrow{B''C''}$ is parallel to $\overleftrightarrow{BC}$. Now apply the parallel projection theorem.)

**4)** (#14, Chapter 5 Exercises) The fundamental theorem on similar triangles allows the trigonometric functions such as sine and cosine to be defined. Namely, given an acute angle $\sphericalangle A$, make it part of a right triangle $\triangle BAC$ with right angle at $C$ and set

$$\sin \sphericalangle A = (\overline{BC})/(\overline{AB})$$

$$\cos \sphericalangle A = (\overline{AC})/(\overline{AB}).$$

These definitions are then independent of the choice of the right triangle used. If $\sphericalangle A$ is obtuse and $\sphericalangle A'$ is its supplement, set

$$\sin \sphericalangle A = +\sin \sphericalangle A'$$

$$\cos \sphericalangle A = -\cos \sphericalangle A'.$$

If $\sphericalangle A$ is a right angle, set

$$\sin \sphericalangle A = 1$$

$$\cos \sphericalangle A = 0.$$

Now, given any triangle $\triangle ABC$, if $a$ and $b$ are the lengths of the sides opposite $A$ and $B$, respectively, prove the law of sines,

$$\frac{a}{b} = \frac{\sin \sphericalangle A}{\sin \sphericalangle B}.$$

(Hint: Drop altitude CD and use the two right triangles $\triangle ADC$ and $\triangle BDC$ to show that $b \sin \sphericalangle A = \overline{CD} = a \sin \sphericalangle B$; see figure 5.16.)

Similarly, prove the law of cosines,

$$c^2 = a^2 + b^2 - 2ab \cos \sphericalangle C,$$

and deduce the converse to the Pythagorean theorem.

**5)** (#1, Chapter 6 Exercises) **This is perhaps the most important exercise in this book.** It is a payoff for all the work you have done. Come back to this exercise as you do subsequent exercises and read further in the book. Your assignment in this exercise is to make a long list of geometric statements that are equivalent to the Euclidean parallel postulate in the sense that they hold in real Euclidean planes and do not hold in real hyperbolic planes. The statements proved in neutral geometry are valid in both Euclidean and hyperbolic planes, so ignore them.

To get you started, Greenberg lists 10 statements that qualify. *They do not say anything about parallel lines*, so you might have been surprised before studying this subject that they are equivalent to the Euclidean parallel postulate. See the book for these statements.

**6)** (#4, Chapter 6 Exercises) Assume that the parallel lines $l$ and $l'$ have a common perpendicular segment $MM'$. Let $A$ and $B$ be any points of $l$ such that $M * A * B$ and drop perpendiculars $AA'$ and $BB'$ to $l'$. Prove that $AA' < BB'$. (Hint: Use Proposition 4.13; see Figure 6.16.)

**7)** (#P-1, Chapter 7 P-Exercises) Using the glossary for the Poincaré disk model, translate the following theorems in hyperbolic geometry into theorems in Euclidean geometry:

a) If two triangles are similar, then they are congruent.

b) If two lines are divergently parallel, then they have a common perpendicular and the latter is unique.

c) The fourth angle of a Lambert quadrilateral is acute.