Name:

Math 331 Final

December 19, 2012

1) (9 points) Give EXACTLY THREE statements that are equivalent in neutral geometry to Hilbert's Euclidean Parallel Postulate: given any line ℓ and any point P not on line ℓ, there exists a unique line m through P parallel to ℓ.

2) Each of the following terms in neutral geometry contains an adjective (first word) and a noun (second word). Define the adjective as it modifies the noun (one of them is a trick):

a) (4 points) convex quadrilateral

b) (3 points) parallel lines

c) (3 points) isosceles triangle

d) (3 points) congruent angles

3) a) (9 points) State the three incidence axioms.

b) (6 points) State EXACTLY TWO of the betweenness axioms.

c) (10 points) State EXACTLY THREE of the congruence axioms.

4) a) (8 points) State the crossbar theorem. Draw a picture to support the validity of this theorem.

b) (6 points) Which of the following ARE conditions for triangle congruence in neutral geometry: SSS, SAS, SSA, ASA, AAS, AAA

5) a) (10 points) Next to each step of the following proof, give the justification for the validity of the step. If your justification is an axiom, you are free to describe the axiom instead of writing something like "Betweenness Axiom 3."

Claim: For every line l and every point P not on l, there exists a line through P that is perpendicular to l.

Proof:

1. Let A and B be any two points on l.

2. On the side of l opposite from P there exists a ray $\overrightarrow{AX}$ such that $\angle XAB \cong \angle PAB$.

3. There exists a point P' on $\overrightarrow{AX}$ such that $AP' \cong AP$.

4. Segment PP' intersects l at a point Q.

5. If $Q = A$, then line $\overleftrightarrow{PP'} \perp l$, and we're done. (Justify carefully!)

6. If $Q \neq A$, then $\triangle PAQ \cong \triangle P'AQ$.

7. Thus, $\angle PQA \cong \angle P'QA$.

8. Thus, $\overleftrightarrow{PP'} \perp l$.

b) (5 points) Draw a picture reflecting the proof in the case where $Q \neq A$.

6) a) (6 points) Define what it means to be a "model" of incidence geometry.

b) (12 points) Let S be the Euclidean sphere in $\mathbb{R}^3$, i.e.,

$$S = \{(x, y, z)|\ x^2 + y^2 + z^2 = 1\}$$

Define a "point" on S to be any point in $\mathbb{R}^3$ satisfying the equation for S. Define a "line" to be any great circle on S, i.e., all points of intersection of S with a plane through $(0, 0, 0)$. Define a point to be "incident" to a line if it satisfies the equations determining the great circle. Is this a model for incidence geometry? Why or why not?

7) (9 points) Let $S = \{1, 2, 3\}$. Define "points" to be one-element subsets of S and "lines" to be two element subsets of S. Define "incidence" to be set containment. S is a model of incidence geometry. Show that S satisfies the spherical parallel postulate: there are no parallel lines.

8) a) (3 points) State the acute angle hypothesis.

b) (3 points) According to Saccheri's Angle Theorem, what property for triangles is equivalent to the acute angle hypothesis?

c) (3 points) For which type of geometry does the acute angle hypothesis hold?

9) (18 points) Prove the Universal Non-Euclidean Theorem: In a Hilbert plane in which rectangles do not exist, for every line ℓ and every point P not on ℓ, there are at least two parallels to ℓ through P.

10) a) (5 points) State the Exterior Angle Theorem.

b) (3 points) According to the Measurement Theorem, if $\overrightarrow{AC}$ is interior to $\sphericalangle DAB$, then what can you say about the angle measures of $\sphericalangle DAB$, $\sphericalangle DAC$, and $\sphericalangle BAC$? It might help to draw a picture.

c) (12 points) Draw two pictures, then use the exterior angle theorem, to show the following result is true: if $\triangle ABC$ has a right or obtuse angle, then the other two angles are acute. WARNING: you are not allowed to use angle measures!

11) a) (12 points) Show that in a Euclidean Hilbert plane, the angle measure of an exterior angle of a triangle is equal to the sum of the angle measures of the remote interior angles.

b) (8 points) Explain why this result is false in Hyperbolic geometry.

12) In neutral geometry, we define a *rhombus* to be a quadrilateral $\square ABCD$ such that all sides are congruent. Let $\square ABCD$ be a rhombus.

a) (10 points) Assuming $\square ABCD$ is convex, prove that opposite sides of a rhombus are parallel, i.e., every rhombus is a parallelogram.

b) (10 points) Prove that $\square ABCD$ is convex.

BONUS: (10 points) Prove the exterior angle theorem.