Exam 1 W '23 2:02

1 a) No, only 0, 1, or infinitely many solutions

b) $$A = \begin{bmatrix} -3 & 9 & -27 \\ -2 & 4 & -8 \\ -1 & 1 & -1 \end{bmatrix}$$

2) a) A is 2×2

B is 4×2

C is 3×2

v is 3×1

b) i) cannot be done

ii) can be done, 2×1

iii) can be done, 3×2

iv) can be done, 4×3

3) a) $y = ax^2 + bx + c$

$(1,-7)$: $-7 = a + b + c$ ①

$(-4,3)$: $3 = 16a - 4b + c$ ②

$(-2,-5)$: $-5 = 4a - 2b + c$ ③

b) ② − ①: $3 = 16a - 4b + c$

$-(-7 = a + b + c)$

$10 = 15a - 5b$

$\rightarrow \quad 2 = 3a - b$

③ − ①: $-5 = 4a - 2b + c$

$-(-7 = a + b + c)$

$2 = 3a - 3b$

$2 = 3a - b$ subtract

$2 = 3a - 3b$

$0 = -2b$, $b = 0$

$3a = 2$,

$a = 2/3$

$-7 = a + b + c$

$-7 = 2/3 + c$

$c = -23/3$

— or —

$$\begin{bmatrix} 1 & 1 & 1 & -7 \\ 1 & -4 & 16 & 3 \\ 1 & -2 & 4 & -5 \end{bmatrix}$$

$-\frac{616}{9} + 107$

$140 \cdot \left(\frac{-44}{90}\right)$

$= \frac{14 \cdot (-44)}{9}$

R2 - R1, R3 - R1

$$\begin{bmatrix} 1 & 1 & 1 & -7 \\ 0 & -5 & 15 & 10 \\ 0 & -3 & 3 & 2 \end{bmatrix}$$

R2 / -5

$$\begin{bmatrix} 1 & 1 & 1 & -7 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -7 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 3 & 2 \end{bmatrix}$$

$3R2 + R3$

$$\begin{bmatrix} 1 & 1 & 1 & -7 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & -6 & -4 \end{bmatrix}$$

$R3/-6$ $$\begin{bmatrix} 1 & 1 & 1 & -7 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & 2/3 \end{bmatrix}$$

$3R3 + R2$

$$\begin{bmatrix} 1 & 1 & 1 & -7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 \end{bmatrix}$$

$-R2 + R1$

$-R3 + R1$

$$\begin{bmatrix} 1 & 0 & 0 & -7-2/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 \end{bmatrix}$$

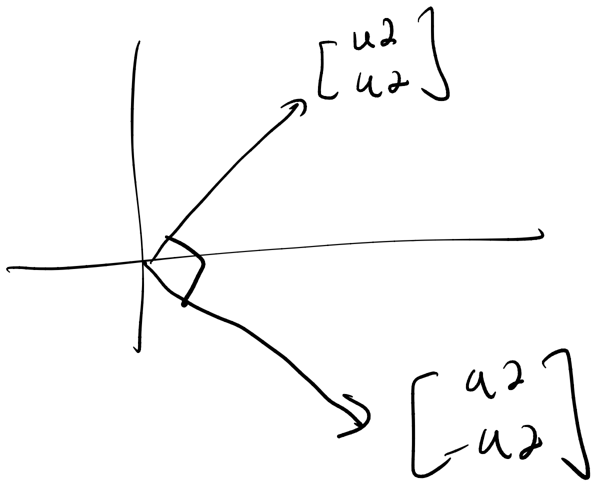
$a = -23/3, \quad c = -2/3, \quad b = 0$

c) $y = -\frac{23}{3}x^2 + 2/3$

4) a) a line

b) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

c)



$$\begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

check: $\begin{bmatrix} 42 \\ 42 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \end{bmatrix} = 210$

$$\frac{210}{42\sqrt{2}\cdot 5} = \frac{1}{\sqrt{2}}$$

$\cos\theta = \frac{1}{\sqrt{2}}$, $\theta = 45^\circ$, 135°

$\begin{bmatrix} 42 \\ -42 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \end{bmatrix} = 210$

$\frac{210}{42\sqrt{2} \cdot 5} = \frac{1}{\sqrt{2}}$

$\cos\theta = \frac{1}{\sqrt{2}}$, $\theta = 45^\circ$, 135°

so 45° will work

d) No – the span of two nonzero vectors is either a line or a plane, and it is a line only when the vectors are multiples.

Since $\vec{v}$ is not a multiple of $\vec{w}$, the span is a plane, which means it must be all of $\mathbb{R}^2$.

5) a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$a^2+bc = a$

$ab+bd = b \rightarrow$ if $b \neq 0$,

$a+d = 1$ ✓

if $b = 0$, $a^2 = a$, so

$a = 0$ or 1

$bc + d^2 = d$

then $d^3 = d$, $d = 0$ or 1

so if $a = d = 1$, $a + d = 2$

if $a = d = 0$, $a + d = 0$

b) $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} (a^2 + c^2) & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(a^2 + c^2)x + (ab + cd)y = 0$

$(ab + cd)x + (b^2 + d^2)y = 0$

Now if $x \neq 0 \neq y$,

$$x\left((a^2+b^2)x+(ab+cd)y\right)=0$$

$$y\left((c^2+d^2)y+(ab+dc)x\right)=0$$

add

$$0 = x^2(a^2+b^2)+2(ab+cd)(xy)+y^2(c^2+d^2)$$

$$\|Ax\|_2^2 = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \cdot \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

$$= (ax+by)^2+(cx+dy)^2$$

$$= (a^2+c^2)x^2+(b^2+d^2)y^2$$

$$+2(ab+cd)xy = 0$$

$$\vec{v}^t A^t \cdot A\vec{v} = \vec{0}$$

$$(A \cdot \vec{v})^t \cdot (A\vec{v}) = \vec{0}$$

so $\|A\vec{v}\|_2^2 = 0$

which means $A\vec{v} = 0$.