Name:

Math 227 Exam 2

March 16, 2023

Directions:

1. WRITE YOUR NAME ON THIS TEST!
2. Except where indicated, merely finding the answer to a problem is not enough to receive full credit; you must show how you arrived at that answer.
3. Unless otherwise indicated, decimal approximations for a numerical answer accurate to 4 decimal places are acceptable.
4. If you have a question, raise your hand or come up and ask me.

1) Let V, W be vector spaces and let $T : V \to W$ be a linear operator.

a) What are the two operations on V, i.e., what makes a vector space?

b) Let $V = M_4(\mathbb{R})$. What are the vectors?

c) If an $n \times n$ matrix A has its determinant equal to zero, what does this tell you regarding the invertibility of A?

d) Which vector space is $\text{Ran}(T)$ a subspace of?

e) Which vector space is $\ker(T)$ a subspace of?

2) Find a single 3×3 matrix that, in homogeneous coordinates,

a) shifts a 2-vector down 15 units and right 4 units,

b) scales the x-coordinate of a 2-vector down by a factor of 7 and scales the y-coordinate down by a factor of 2,

c) rotates a 2-vector by $11\pi/6$ radians counterclockwise,

d) If A, B, and C are the matrices from parts a), b), and c), respectively, in what order do you write the product of A, B, and C if you first rotate, then shift, then scale?

3) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 2y \\ x - 3y \end{bmatrix}$$

a) Find a matrix representation A for T.

b) Is A invertible? Why or why not?

c) Find the area of the parallelogram with vertices $(0, 0)$, $(2, 1)$, $(2, -3)$, and $(4, -2)$. Be sure to draw a picture!

4) Recall that $\mathcal{S}$ is the vector space of all sequences of real numbers. Let $W \subseteq \mathcal{S}$,

$$W = \{(a_n)_{n=1}^{\infty} \mid a_1^2 + a_3^2 = 0\}$$

a) Write down two sequences in W.

b) Write down a sequence that is NOT in W (if possible).

c) Show that W is a subspace of $\mathcal{S}$.

5) Let $V = W = M_2(\mathbb{R})$ and define $T : V \to W$,

$$T(A) = \frac{A + A^t}{2},$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$.

a) Write down two nonzero vectors in $\ker(T)$.

b) Write down a nonzero vector in $\mathrm{Ran}(T)$.

c) Determine $\ker(T)$, with reasoning to support your assertion.

BONUS: (10 points) For every invertible matrix $A \in M_n(\mathbb{R})$, show that $\mathrm{Ran}(A) = \mathbb{R}^n$.