Fall 2019 Exam 3

1) a) $\det(A - \lambda I)$

$$= \det\left(\begin{bmatrix} 14-\lambda & -9 \\ 18 & -13-\lambda \end{bmatrix}\right)$$

$$= \lambda^2 - \lambda - 182 + 162$$

$$= \lambda^2 - \lambda - 20$$

$$= (\lambda - 5)(\lambda + 4)$$

$$\lambda = 5, -4$$

b) $\lambda = 5$ $\quad$ $\lambda = -4$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

c) $A\begin{bmatrix}1\\1\end{bmatrix}$

$$=\begin{bmatrix}14 & -9\\18 & -13\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$=\begin{bmatrix}5\\5\end{bmatrix}=5\begin{bmatrix}1\\1\end{bmatrix} \checkmark$$

$A\begin{bmatrix}1\\2\end{bmatrix}$

$$=\begin{bmatrix}14 & -9\\18 & -13\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$$

$$=\begin{bmatrix}-4\\-8\end{bmatrix}=-4\begin{bmatrix}1\\2\end{bmatrix} \checkmark$$

2) a) $y = ax^2 + bx + c$

$$2 = c$$

$$4 = 9a - 3b + c$$

$$8 = 25a + 5b + c$$

$$-16 = 81a + 9b + c$$

b) $$\begin{bmatrix} 0 & 0 & 1 \\ 9 & -3 & 1 \\ 25 & 5 & 1 \\ 81 & 9 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \\ -16 \end{bmatrix}$$

c) $$A^tA = \begin{bmatrix} 7267 & 827 & 115 \\ 827 & 115 & 11 \\ 115 & 11 & 4 \end{bmatrix}$$

$$A^tb = \begin{bmatrix} -1060 \\ -116 \\ -2 \end{bmatrix}$$

d) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^t A)^{-1} A^t b$

$$= \begin{bmatrix} -\frac{2377}{6234} \\ \frac{6197}{6234} \\ \frac{8030}{1039} \end{bmatrix}$$

$$y = -\frac{2377}{6234} x^2 + \frac{6197}{6234} x + \frac{8030}{1039}$$

$$\approx -.3813 x^2 + .9941 x + 7.7286$$

3) a) $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}8\\-42\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\13\end{bmatrix}$

$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\-7\end{bmatrix}$

$A = \begin{bmatrix}8 & 1 & 5\\-42 & 13 & -7\end{bmatrix}$

b) $\begin{bmatrix}8\\-42\end{bmatrix}, \begin{bmatrix}1\\13\end{bmatrix}, \begin{bmatrix}5\\-7\end{bmatrix}, \begin{bmatrix}6\\6\end{bmatrix}$

c) From Wolfram Alpha,

$\ker(T) = \left\{\begin{bmatrix}-29x\\-77x\\31x\end{bmatrix} \mid x \in \mathbb{R}\right\}, \begin{bmatrix}-29\\-77\\31\end{bmatrix}$

d) $P = uu^t$ where u is a unit vector in $\text{Ran}(P)$.

$$v = \begin{bmatrix} -29 \\ -77 \\ 31 \end{bmatrix}$$

$$\|v\|_2 = 3\sqrt{859} \approx 87.9261$$

$$P = 9 \cdot 859 \cdot \begin{bmatrix} -29 \\ -77 \\ 31 \end{bmatrix} \begin{bmatrix} -22 & -77 & 31 \end{bmatrix}$$

$$= 7731 \begin{bmatrix} 841 & 2233 & -899 \\ 2233 & 5929 & -2387 \\ -899 & -2387 & 961 \end{bmatrix}$$

4) a) Let $f, g \in \mathcal{F}(\mathbb{R})$, $x \in \mathbb{R}$, $c \in \mathbb{R}$

$$
\begin{aligned}
T(f+g)(x) &= x^2 (f+g)(x) \\
&= x^2 f(x) + x^2 g(x) \\
&= T(f)(x) + T(g)(x)
\end{aligned}
$$

$$
\begin{aligned}
T(cf)(x) &= x^2 (cf)(x) \\
&= c x^2 f(x) \\
&= c T(f)(x)
\end{aligned}
$$

5) $P^2 = P \quad Q^2 = Q \quad P = P^t \quad Q = Q^t$

$$(P+Q)^t = P^t + Q^t = P + Q \checkmark$$

$$(P+Q)(P+Q) = P^2 + Q^2 + PQ + QP$$

$PQ = 0$ and

$QP = Q^t P^t = (PQ)^t = 0$

So $(P+Q)(P+Q) = P^2 + Q^2 = P + Q \checkmark$