Final Fall 18 Sol

$$\begin{bmatrix} -2 & 6 & -1 & 3 \\ 3 & -11 & 4 & 5 \\ 9 & 6 & -8 & 31 \end{bmatrix}$$

$-3R2 + R3$

$$\begin{bmatrix} -2 & 6 & -1 & 3 \\ 3 & -11 & 4 & 5 \\ 0 & 39 & -20 & 16 \end{bmatrix}$$

$R2 + R1$

$$\begin{bmatrix} 1 & -5 & 3 & 8 \\ 3 & -11 & 4 & 5 \\ 0 & 39 & -20 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 3 & 8 \\ 3 & -11 & 4 & 5 \\ 0 & 39 & -20 & 16 \end{bmatrix}$$

$-3R1 + R2$

$$\begin{bmatrix} 1 & -5 & 3 & 8 \\ 0 & 4 & -5 & -19 \\ 0 & 39 & -20 & 16 \end{bmatrix}$$

$R2/4$

$$\begin{bmatrix} 1 & -5 & 3 & 8 \\ 0 & 1 & -5/4 & -19/4 \\ 0 & 39 & -20 & 16 \end{bmatrix}$$

$5R2 + R1$

$$\begin{bmatrix} 1 & 0 & -13/4 & -63/4 \\ 0 & 1 & -5/4 & -19/4 \\ 0 & 39 & -20 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -13/4 & -63/4 \\ 0 & 1 & -5/4 & -19/4 \\ 0 & 39 & -20 & 16 \end{bmatrix}$$

$-39R2 + R3$

$$\begin{bmatrix} 1 & 0 & -13/4 & -63/4 \\ 0 & 1 & -5/4 & -19/4 \\ 0 & 0 & \frac{115}{4} & \frac{805}{4} \end{bmatrix}$$

$\frac{4}{115} R3$

$$\begin{bmatrix} 1 & 0 & -13/4 & -63/4 \\ 0 & 1 & -5/4 & -19/4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -13/4 & -63/4 \\ 0 & 1 & -5/4 & -19/4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$13/4 R3 + R1$; $5/4 R3 + R2$

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$x = 7$, $y = 4$, $z = 7$

2) a) $0 = \det \begin{bmatrix} -168-\lambda & 110 \\ -264 & 173-\lambda \end{bmatrix}$

$= \lambda^2 - 5\lambda - 29064 + 29040$

$= \lambda^2 - 5\lambda - 24$

$= (\lambda - 8)(\lambda + 3)$

$\lambda = 8, -3$

b) $\lambda = 8$: $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$\lambda = -3$: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

c)
$$\begin{bmatrix} -168 & 110 \\ -264 & 173 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} -840+880 \\ -1320+1384 \end{bmatrix} = \begin{bmatrix} 40 \\ 64 \end{bmatrix} = 8 \begin{bmatrix} 5 \\ 8 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} -168 & 110 \\ -264 & 173 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -336+330 \\ -528+519 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \end{bmatrix}$$

3) a) $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 42 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} \cos(\pi) & -\sin(\pi) & 0 \\ \sin(\pi) & \cos(\pi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a) ACB

$$= \begin{bmatrix} -8 & 0 & -48 \\ 0 & -8 & -336 \\ 0 & 0 & 1 \end{bmatrix}$$

4) $ax^2+bx+c=y$

a) $a+b+c=4$

$9a-3b+c=6$

$a-b+c=7$

b) $$\mathrm{RREF}\begin{bmatrix} 1 & 1 & 1 & 4 \\ 9 & -3 & 1 & 6 \\ 1 & -1 & 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$$y = -\frac{x^2}{2} - \frac{3}{2}x + 6$$

5) a) $y = mx + b$

$m + b = 4$

$-3m + b = 6$

$-m + b = 7$

b) $\begin{bmatrix} 1 & 1 \\ -3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}$

c) $A^t A = \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$

$A^t b = \begin{bmatrix} -21 \\ 17 \end{bmatrix}$

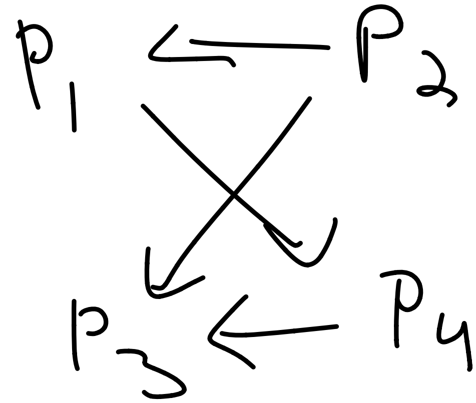
$\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -21 \\ 17 \end{bmatrix}$

d) $(A^t A)^{-1} A^t b$

$$= \begin{bmatrix} -1/2 \\ 31/6 \end{bmatrix}$$

$$y = -1/2\, x + 31/6$$

6) a)



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

b) $$A' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1/2 & 1/4 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/2 & 1/4 & 1 \\ 1 & 0 & 1/4 & 0 \end{bmatrix}$$

c) $\frac{17}{20} B + \frac{3/20}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$$= \frac{1}{80} \begin{bmatrix} 3 & 37 & 20 & 3 \\ 3 & 3 & 20 & 3 \\ 3 & 37 & 20 & 71 \\ 71 & 3 & 20 & 3 \end{bmatrix}$$

d) $\lambda = 1$

e) Page Rank of P_2

$$= \left(800/1769\right) \cdot 1 \Big/ \left(\frac{127053}{35380}\right)$$

$$= \frac{1600}{127053} \approx .125932$$

4)

a)
$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

b)
$$R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 80 & 0 & 0 \\ 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 40 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 80 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 60 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 40 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ V_1 \\ V_2 \\ V_3 \\ V \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

d) $I_1 = I_2 = I_3 = I_4 = 4/45$

$$v_1 - v_4 = -59/9$$

$$v_2 - v_4 = -8/9$$

$$v_3 - u_4 = -16/3$$

8) a) $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

b) $v = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

$r = \sqrt{3^2 + 6^2} = \sqrt{45}$

$\theta = \arctan\left(\frac{-3}{6}\right)$

$\approx .463648$ radians

$\approx -26.5651^\circ$

$$v_1 = \begin{bmatrix} \cos(\theta + 21) \\ \sin(\theta + 21) \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \cos(\theta - 21) \\ \sin(\theta - 21) \end{bmatrix}$$

$$v_1 \approx \begin{bmatrix} .995287 \\ -.0969758 \end{bmatrix}$$

$$v_2 \approx \begin{bmatrix} .674753 \\ -.738044 \end{bmatrix}$$

— or —

$$w = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with:}$$

$$x^2 + y^2 = 1 \quad \text{and}$$

$$\frac{v \cdot w}{\|v\|} = \cos(21)$$

$$\cos(21) = \frac{v \cdot w}{\|v\|} = \frac{6x - 3y}{\sqrt{45}}$$

$$\sqrt{45}\cos(21) = 6x - 3y$$

$$3y = 6x - \sqrt{45}\cos(21)$$

$$y = 2x - \sqrt{5}\cos(21)$$

$$x^2 + (2x - \sqrt{5}\cos(21))^2 = 1$$

$x \approx .915287$

or

$.674753$

$y = 2x - \sqrt{5}\cos(21)$

$\approx -.0969758$

or

$-.738044$

7) a) $\begin{bmatrix} 94 & 33 \\ 33 & 77 \end{bmatrix}$

b) $D = \frac{1}{2}\begin{bmatrix} 171-\sqrt{4645} & 0 \\ 0 & 171+\sqrt{4645} \end{bmatrix}$

$$O = \frac{1}{\sqrt{4645-17\sqrt{4645}}}\begin{bmatrix} 17-\sqrt{4645} & 17+\sqrt{4645} \\ 66 & 66 \end{bmatrix}$$

$$A^tA = ODO^t$$

c) Singular values

$$\sqrt{\frac{1}{2}\left(16943+171\sqrt{4645}\right.},$$

$$\sqrt{\frac{1}{2}\left(16943-171\sqrt{4645}\right.}$$

10) a) infinite sequences

b) i) zero vector

$(0,0,0 \ldots)$

$c=0$ ✓

ii) Let $(a_i)_{i=1}^{\infty}, (b_i)_{i=1}^{\infty} \in W$

$\alpha \in \mathbb{R}$

Then $\exists\ c_1, c_2\ \ a_i = c_1$

$b_i = c_2 \ \forall\ i \geq 1$

$$\alpha (a_i)_{i=1}^{\infty} + (b_i)_{i=1}^{\infty}$$

$$= \alpha (c_1, c_1, c_1, \ldots) + (c_2, c_2, c_2, \ldots)$$

$$= (\alpha c_1, \alpha c_1, \alpha c_1, \ldots) + (c_2, c_2, c_2, \ldots)$$

$$= (\alpha c_1 + c_2, \alpha c_1 + c_2, \alpha c_1 + c_2, \ldots)$$

$$c = \alpha c_1 + c_2 \checkmark$$

— or —

$$W = \{c(1,1,1,\ldots,1) \mid c \in \mathbb{R}\}$$

$$= \operatorname{span}\{(1,1,1,\ldots,1)\}$$

which is a subspace

(or just characterize as multiples of a given vector)

11) $T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}2\\-11\\-1\end{bmatrix}$

$T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-3\\8\\-7\end{bmatrix}$

$T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\0\\5\end{bmatrix}$

$$A=\begin{bmatrix}2 & -3 & 1\\-11 & 8 & 0\\-1 & -7 & 5\end{bmatrix}$$

b) ker(T)

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\text{RREF} \left[A \;\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right]$$

$$= \begin{bmatrix} 1 & 0 & -8/17 & 0 \\ 0 & 1 & -11/17 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \frac{8}{17} z, \quad y = 11/17\, z$$

Basis

$$\begin{bmatrix} 8 \\ 11 \\ 17 \end{bmatrix}$$

12) a) $\text{Ran}(T)$

$$= \text{span}\left\{ \begin{bmatrix} 2 \\ -11 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \right\}$$

$\text{RREF}(A^t)$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So only 2 vectors needed

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

b) $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{3\sqrt{6}} \begin{bmatrix} 2 \\ -5 \\ 5 \end{bmatrix} \right\}$

c) $v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$v_2 = \frac{1}{3\sqrt{6}} \begin{bmatrix} 2 \\ -5 \\ 5 \end{bmatrix}$

$$P = v_1 v_1^t + v_2 v_2^t$$

$$= \frac{1}{27} \begin{bmatrix} 2 & -5 & 5 \\ -5 & 26 & 1 \\ 5 & 1 & 26 \end{bmatrix}$$