Final Fall '21

1) a) no, only zero, 1, or infinitely many solutions are possible

b) addition and scalar multiplication

c) point, lines, plane ($\mathbb{R}^2$)

d) point, lines, planes, space ($\mathbb{R}^3$)

e) matrix

2) a) 2×1

b) no

c) 2×2

d) 4×4

e) 4×2

3) $y = ax^2 + bx + c$

a) $-3 = 16a + 4b + c$

$3 = 4a + 2b + c$

$2 = 4a - 2b + c$

b) $$\left[\begin{array}{ccc|c} 1 & 4 & 16 & -3 \\ 1 & 2 & 4 & 3 \\ 1 & -2 & 4 & 2 \end{array}\right]$$

$$\underset{\substack{-R1+R2 \\ -R1+R3}}{\longmapsto} \begin{bmatrix} 1 & 4 & 16 & -3 \\ 0 & -2 & -12 & 6 \\ 0 & -6 & -12 & 5 \end{bmatrix}$$

$$\underset{-3R2+R3}{\longmapsto} \begin{bmatrix} 1 & 4 & 16 & -3 \\ 0 & -2 & -12 & 6 \\ 0 & 0 & 24 & -13 \end{bmatrix}$$

$\frac{1}{2}R3 + R2 \mapsto$
$$\begin{bmatrix} 1 & 4 & 16 & -3 \\ 0 & -2 & 0 & -1/2 \\ 0 & 0 & 24 & -13 \end{bmatrix}$$

$2R2 + R1 \mapsto$
$$\begin{bmatrix} 1 & 0 & 16 & -4 \\ 0 & -2 & 0 & -1/2 \\ 0 & 0 & 24 & -13 \end{bmatrix}$$

$-2/3 R3 + R1$
$$\begin{bmatrix} 1 & 0 & 0 & 14/3 \\ 0 & -2 & 0 & -1/2 \\ 0 & 0 & 24 & -13 \end{bmatrix}$$

$R2/-2,\ R3/24$
$$\begin{bmatrix} 1 & 0 & 0 & 14/3 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -13/24 \end{bmatrix}$$

c) $-13/24\, x^2 + 1/4\, x + 14/3 = y$

4)

a) $\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} \cos(7\pi/6) & -\sin(7\pi/6) & 0 \\ \sin(7\pi/6) & \cos(7\pi/6) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -\frac{\sqrt{3}}{2} & 1/2 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d) $B \cdot A \cdot C$

5)

$$y = mx + b$$

a) $-2 = -7m + b$

$-7 = 5m + b$

$-5 = -9m + b$

b) $\begin{bmatrix} 1 & -7 \\ 1 & 5 \\ 1 & -9 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \\ -5 \end{bmatrix}$

c) $A^t \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ -7 & 5 & -9 \end{bmatrix} \begin{bmatrix} 1 & -7 \\ 1 & 5 \\ 1 & -9 \end{bmatrix}$

$$= \begin{bmatrix} 3 & -11 \\ -11 & 155 \end{bmatrix}$$

$$A^t \cdot \vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ -7 & 5 & -9 \end{bmatrix} \begin{bmatrix} -2 \\ -7 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -14 \\ 24 \end{bmatrix}$$

d)

$$\det \begin{bmatrix} 3 & -11 \\ -11 & 155 \end{bmatrix} = 465 - 121$$

$$= 344 \neq 0$$

so invertible

$$\begin{bmatrix} 3 & -11 \\ -11 & 155 \end{bmatrix}^{-1} = \frac{1}{344} \begin{bmatrix} 155 & 11 \\ 11 & 3 \end{bmatrix}$$

$$\underbrace{\frac{1}{344}\begin{bmatrix} 155 & 11 \\ 11 & 3 \end{bmatrix}\begin{bmatrix} -14 \\ 24 \end{bmatrix}} = \begin{bmatrix} b \\ m \end{bmatrix}$$

$$= \frac{1}{172}\begin{bmatrix} -953 \\ -41 \end{bmatrix}$$

$$y = -\frac{41}{172}x - \frac{953}{172}$$

$$\approx -.238372x - 5.5407$$

6) a) $\begin{bmatrix} 2 & -2 & -5 \\ 0 & 5 & 0 \\ -1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$$= \begin{bmatrix} 2-\lambda & -2 & -5 \\ 0 & 5-\lambda & 0 \\ -1 & 0 & -2-\lambda \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 2-\lambda & -2 & -5 \\ 0 & 5-\lambda & 0 \\ -1 & 0 & -2-\lambda \end{bmatrix}\right)$$

$$= \begin{array}{ccccc} 2-\lambda & -2 & -5 & 2-\lambda & -2 \\ 0 & 5-\lambda & 0 & 0 & 5 \\ -1 & 0 & -2-\lambda & -1 & 0 \end{array}$$

$= (2-\lambda)(5-\lambda)(-2-\lambda) - 5(5-\lambda)$

$= (5-\lambda)\left((2-\lambda)(-2-\lambda) - 5\right)$

$= (5-\lambda)(\lambda^2 - 4 - 5)$

$= (5-\lambda)(\lambda-3)(\lambda+3)$

$\lambda = 3, -3, 5$

b) $A \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 & -5 \\ 0 & 5 & 0 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} = -3 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ✓

$$A \cdot \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -5 \\ 0 & 5 & 0 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 \\ 0 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \checkmark$$

$$A \cdot \begin{bmatrix} 7 \\ -8 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -5 \\ 0 & 5 & 0 \\ -1 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -8 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 35 \\ -40 \\ -5 \end{bmatrix} = 5 \cdot \begin{bmatrix} 7 \\ -8 \\ -1 \end{bmatrix} \checkmark$$

7) a) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/3 & 0 & 1/2 & 0 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 1/2 & 1/2 & 0 \end{bmatrix}$

c) $17/20\, B + \dfrac{3/20}{4} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3/80 & 3/80 & 3/80 & 71/80 \\ 77/240 & 3/80 & 37/80 & 3/80 \\ 77/240 & 37/80 & 3/80 & 3/80 \\ 77/240 & 37/80 & 37/80 & 3/80 \end{bmatrix}$

d) $$\frac{4389}{4389+4287+2\cdot(3080)} = \frac{4389}{14836} \approx .2958$$

8) a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 1 \\ -2 & -4 \\ 1 & 3 \end{bmatrix}$

b) $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$

Ran(T) is a plane

c) $A^t \cdot A = \begin{bmatrix} -1 & -2 & 1 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -2 & -4 \\ 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 10 \\ 10 & 26 \end{bmatrix}$$

$\det(A^tA) = 26 \cdot 6 - 100 = 156 - 100$

$= 56 \neq 0$

So $P = A \cdot (A^t \cdot A)^{-1} \cdot A^t$

4) a)

$$\text{RREF} \begin{bmatrix} 2 & 4 & 0 \\ -1 & 0 & 2 \\ 3 & 1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $-2 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}$, and

v_3 is in the span of v_1 and v_2

b) $v_1 = w_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$$w_2 = v_2 - \frac{v_2^t \cdot v_1}{v_1^t \cdot v_1} v_1$$

$$v_2^t \cdot v_1 = \begin{bmatrix} 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= 11$$

$$v_1^t \cdot v_1 = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= 14$$

$$w_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} - \frac{11}{14} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 11/7 \\ -11/14 \\ 33/14 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 17/7 \\ 11/14 \\ -19/14 \end{bmatrix}$$

could take $14\, w_2$

$$= \begin{bmatrix} 34 \\ 11 \\ -19 \end{bmatrix}$$

$$u_1 = \frac{v_1}{\|v_1\|_2} = \frac{1}{\sqrt{11}} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$u_2 = \frac{w_2}{\|w_2\|_2} = \frac{1}{\sqrt{1638}} \begin{bmatrix} 34 \\ 11 \\ -19 \end{bmatrix}$$

c) want $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{x}$ with

$$\vec{x}^t \cdot v_1 = 0 = \vec{x}^t \cdot v_3$$

$$0 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = 4x + z$$

$$0 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix} = 2y - 5z$$

set $z = 1$, $x = -1/4$, $y = 5/2$

$$\begin{bmatrix} -1/4 \\ 5/2 \\ 1 \end{bmatrix}$$

d) use $u\vec{x} = \begin{bmatrix} -1 \\ 10 \\ 4 \end{bmatrix} = w$

$$\|w\|_2 = \sqrt{117}$$

$$v = \frac{w}{\|w\|_2}$$

$$P = v \cdot v^t$$

$$= \frac{1}{117} \begin{bmatrix} -1 \\ 10 \\ 4 \end{bmatrix} \begin{bmatrix} -1 & 10 & 4 \end{bmatrix}$$

$$= \frac{1}{117} \begin{bmatrix} -1 & 10 & 4 \\ -10 & 100 & 40 \\ -4 & 40 & 16 \end{bmatrix}$$

10) a) No! A could just be non-invertible.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A \cdot \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b) $\lambda = 0$. This is because if $A \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$, then A invertible gives

$$\vec{v} = A^{-1} \cdot \vec{0} = \vec{0},$$

but $\vec{v}$ is nonzero for eigenvectors.

11)

a) zero vector

$(0, 0, 0, \ldots)$

$a_n = 0, \quad a_{n-1} = 0, \quad a_{n-2} = 0,$

So $0 = 0 + 0$ ✓

b) Suppose $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty} \in W$.

Then $a_n = a_{n-1} + a_{n-2}$

$b_n = b_{n-1} + b_{n-2}$

$a_n + b_n = (a_{n-1} + a_{n-2}) + (b_{n-1} + b_{n-2})$

$= (a_{n-1} + b_{n-1}) + (a_{n-2} + b_{n-2})$

So if $c_n = a_n + b_n$

$c_n = c_{n-1} + c_{n-2}$ ✓

c) if $c \in \mathbb{R}$, $(a_n)_{n=1}^{\infty} \in W$, then

$$c \cdot a_n = c \cdot (a_{n-1} + a_{n-2})$$

$$= c \cdot a_{n-1} + c \cdot a_{n-2}$$

if $b_n = c \cdot a_n$,

$$b_n = b_{n-1} + b_{n-2}$$ ✓