Fall '23 final

1) a) No, only one, zero, or infinitely many solutions.

b) vector addition and scalar multiplication

c) point, line, plane ($\mathbb{R}^2$)

d) point, line, plane, 3-D space ($\mathbb{R}^3$)

e) matrix

2) a) yes, 2×1

b) No

c) No

d) No

e) No

3) $y = ax^3 + bx^2 + cx + d$

a) $4 = a + b + c + d$ (1)

$8 = -a + b - c + d$ (2)

$7 = -8a + 4b - 2c + d$ (3)

$5 = d$

b) Add (1) and (2)

$12 = 2b + 2d = 2b + 10$

$2b = 2$, $b = 1$

Subtract (2) from (1)

$-4 = 2a + 2c$

$a + c = -2$

$c = -2 - a$

Plug into 3^{rd} equation

$7 = -8a + 4 - 2(-2-a) + 5$

$7 = -6a + 13$

$-6a = -6$

$a = 1$

$c = -2 - a = -3$

c) $y = x^3 + x^2 - 3x + 5$

4)

a) $\begin{bmatrix} 11 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 16 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) & 0 \\ \sin(\pi/6) & \cos(\pi/6) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d) $A \cdot B \cdot C$

5) $y = mx + b$

a) $4 = m + b$

$8 = -m + b$

$7 = -2m + b$

$5 = b$

b) $$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 7 \\ 5 \end{bmatrix}$$

c) $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} = A^t \cdot A$$

$$A^t \cdot b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 7 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 24 \\ -26 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 24 \\ -26 \end{bmatrix}$$

d)

$$\det\left(\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}\right) = 24 - 4 = 20 \neq 0$$

so

$$\begin{bmatrix} b \\ m \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 24 \\ -26 \end{bmatrix}$$

$$\begin{bmatrix} b \\ m \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 92 \\ -56 \end{bmatrix}$$

$$y = \frac{-56}{20} x + \frac{92}{20}$$

$$y = \frac{-12}{5} x + \frac{23}{5}$$

6) a) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

b) $\det \begin{bmatrix} 8-\lambda & 1 & -4 \\ 0 & -2-\lambda & 0 \\ 5 & 0 & -1-\lambda \end{bmatrix}$

$$= \begin{array}{ccccc} 8-\lambda & 1 & -4 & 8-\lambda & 1 \\ 0 & -2-\lambda & 0 & 0 & -2-\lambda \\ 5 & 0 & -1-\lambda & 5 & 0 \end{array}$$

$$= (8-\lambda)(-2-\lambda)(-1-\lambda) + 20(-2-\lambda)$$

$$= (-2-\lambda)\left(\lambda^2 - 7\lambda - 8 + 20\right)$$

$$= (-2-\lambda)(\lambda^2 - 7\lambda + 12)$$

$$= (-2-\lambda)(\lambda - 4)(\lambda + 3)$$

$\lambda = 4, -3, -2$

7) a) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = A$

b) $A' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 1/4 & 1/3 & 0 \\ 1/2 & 1/4 & 1/3 & 1 \\ 0 & 1/4 & 0 & 0 \\ 1/2 & 1/4 & 1/3 & 0 \end{bmatrix}$

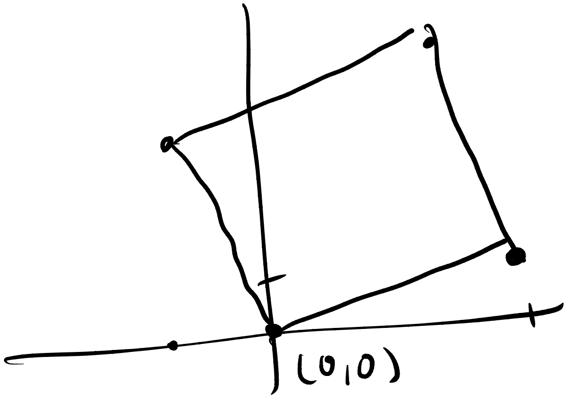
c) $C = \frac{17}{20} B + \frac{3/20}{4} E$

$$C = \begin{bmatrix} 3/80 & 1/4 & 77/240 & 3/80 \\ 37/80 & 1/4 & 77/240 & 71/80 \\ 3/80 & 1/4 & 3/80 & 3/80 \\ 37/80 & 1/4 & 77/240 & 3/80 \end{bmatrix}$$

d) $$\frac{61600}{61600 + 162393 + 48000 + 87780}$$

$\approx$

8) a)



$$A = \begin{bmatrix} 6 & -2 \\ 1 & 4 \end{bmatrix}$$

$$\text{Area} = \left|\det \begin{bmatrix} 6 & -2 \\ 1 & 4 \end{bmatrix}\right| = |24 + 2|$$

$$= 26$$

6)

vector $= \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} 1 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + 4y = 0$$

$$x = -4y$$

$$\begin{bmatrix} 0 & 12 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$12y - 6z = 0$$

$$z = 2y$$

choose $y = 1$

$$\begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}$$

$$\left\| \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix} \right\|_2 = \sqrt{16+4+1}$$

$$= \sqrt{21}$$

$$\frac{7}{\sqrt{21}} \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}$$ or its negative

9) a) $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$

$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\-1\\0\\-2\end{bmatrix}$

$$A=\begin{bmatrix}1 & 0 & -1\\0 & 1 & -1\\1 & -1 & 0\\1 & 1 & -2\end{bmatrix}$$

b) $\mathrm{Ran}(T)=\mathrm{Col}(A)$

$=\mathrm{span}\left(\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}-1\\-1\\0\\-2\end{bmatrix}\right)$

The first 2 vectors aren't multiples.

Then if we solve

$$x\begin{bmatrix}1\\0\\1\\1\end{bmatrix}+y\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}=\begin{bmatrix}-1\\-1\\0\\-2\end{bmatrix}$$

so $x=-1$

$y=-1$

$x-y=0$

$x+y=-2$

with $x=-1=y$, all equations are satisfied

so $\operatorname{Ran}(T)=\operatorname{span}\left(\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\right\}\right)$

= a plane

c) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$x - z = 0 \rightarrow x = z$

$y - z = 0 \rightarrow y = z$

$x - y = 0 \rightarrow x = y$

$x + y - 2z = 0 \rightarrow x + y = 2z$

$x = y = z$

$\ker(T) =$ multiples of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

d) $\left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\|_2 = \sqrt{3}$

$\vec{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$P = \vec{v} \cdot \vec{v}^{\,t} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

10) a) $\begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix} = x\begin{bmatrix} -1 \\ 2 \\ 8 \end{bmatrix} + y\begin{bmatrix} -3 \\ 8 \\ 48 \end{bmatrix}$

$-3y - x = 0 \rightarrow x = -3y$

$2x + 8y = 1$

$8x + 48y = 7$

plugging in 1

$2(-3y) + 8y = 1$

$2y = 1$

$y = 1/2, \quad x = -3/2$

$-3/2 \cdot 8 + \frac{1}{2} \cdot 48 =$

$24 - 12 = 12 \neq 7$

So $\vec{v}_1$ is not in the span of $\vec{v}_2$ and $\vec{v}_3$.

b) $\vec{v}_2$ and $\vec{v}_3$ aren't multiples, so span $(\vec{v}_2, \vec{v}_3, \vec{v}_1)$ is at least a plane. But $\vec{v}_1$ isn't in the span of $\vec{v}_2$ and $\vec{v}_3$ by a), so span$(\vec{v}_2, \vec{v}_3, \vec{v}_1)$ isn't a plane. Therefore, the span must be $\mathbb{R}^3$.

Orthonormal basis for W:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b) Suppose A is not invertible.

Then there is a nonzero

$\vec{v} \in \mathbb{R}^n$ with

$$A\vec{v} = \vec{0}.$$

multiplying by A^t,

$$A^t \cdot A\vec{v} = A^t \cdot \vec{0} = \vec{0}$$

multiplying by $(A^tA)^{-1}$,

$$\vec{0} = (A^tA)^{-1} \cdot \vec{0} = (A^t \cdot A)^{-1} \cdot (A^t \cdot A)\vec{v}$$

$$= \vec{v}$$

But $\vec{v}$ was nonzero, so this can't happen.

(2) a) $f(x)=0$,

$f(x)=1$

$f(x)=2$

b) $f(x)=x$

c) W is nonempty by a).

Now let $f, g \in W$, $k \in \mathbb{R}$.

Then $f(x)=f(-x)$

and $g(x)=g(-x)$ for all $x \in \mathbb{R}$.

So $(f+g)(x)=f(x)+g(x)$

$=f(-x)+g(-x)$

$=(f+g)(-x)$ ✓

Also, $(k \cdot f)(x) = k \cdot f(x)$

$$= k \cdot f(-x)$$

$$= (k \cdot f)(-x)$$

Therefore, W is a subspace of $\mathcal{F}(\mathbb{R})$.