Name:

Math 227 Final

December 14, 2018

Directions:

1. WRITE YOUR NAME ON THIS TEST!
2. Wolfram Alpha or a similar program may be used for all computational problems.
3. Unless otherwise indicated, decimal approximations for a numerical answer accurate to 4 decimal places are acceptable.
4. If you have a question, raise your hand or come up and ask me.
5. If you had a choice on whether to take this exam, please indicate your understanding of the potential consequences by signing the statement(s) below:

   I understand that by taking this exam, I may lower my grade from what it was before the final.

   Signed: ______________________________

   I had a choice on whether to take this exam, and now that I've seen it, I want to keep my pre-final grade!

   Signed: ______________________________

1) (12 points) Solve the following system of equations BY HAND, using any manner at your disposal, or show there is no solution. SHOW YOUR WORK.

$$-2x + 6y - z = 3$$
$$3x - 11y + 4z = 5$$
$$9x + 6y - 8z = 31$$

2) Let

$$A = \begin{bmatrix} -168 & 110 \\ -264 & 173 \end{bmatrix}.$$

a) (8 points) Compute all eigenvalues of A BY HAND.

b) (2 points) Find an associated eigenvector for each eigenvalue from a).

c) (6 points) Check that each actually vector from b) actually is an eigenvector BY HAND.

3) Find the matrix of the linear transformations on $\mathbb{R}^3$ that, in homogeneous coordinates,

a) (3 points) scales a 2-vector up by a factor of 8.

b) (4 points) shifts a 2-vector right 6 units and up 42 units

c) (5 points) rotates a 2-vector by π radians counterclockwise

d) (6 points) shifts a 2-vector up 6 units and right 42 units, then rotates a 2-vector by π radians counterclockwise, and finally scales a 2-vector up by a factor of 8.

4) Find the interpolating quadratic through the points $(1, 4)$, $(-3, 6)$, and $(-1, 7)$ in $\mathbb{R}^2$ by,

a) (8 points) writing down a system of linear equations that determines the coefficients of the polynomial, then

b) (6 points) finding the solution to the equation and writing down the polynomial.

5) Given the points $(1, 4)$, $(-3, 6)$, and $(-1, 7)$ in $\mathbb{R}^2$, find the best-fit line to the points by

a) (6 points) Finding a system of linear equations that represents a "solution" to the problem,

b) (6 points) Writing the problem as a matrix equation $Ax = b$,

c) (5 points) Finding the system $A^t Ax = A^t b$, computing both $A^t A$ and $A^t b$,

d) (5 points) Solving the system in c) and producing the polynomial.

6) Given the simplified link diagram between webpages P_1, P_2, P_3 and P_4 described by

- P_1 links to P_4
- P_2 links to P_1 and P_3
- P_3 doesn't link to anything
- P_4 links to P_3

a) (5 points) Construct the link matrix A.

b) (6 points) Find the normalized matrix B.

c) (11 points) Calculate the PageRank matrix C, using $d = .85 = 17/20$.

d) See the next page!

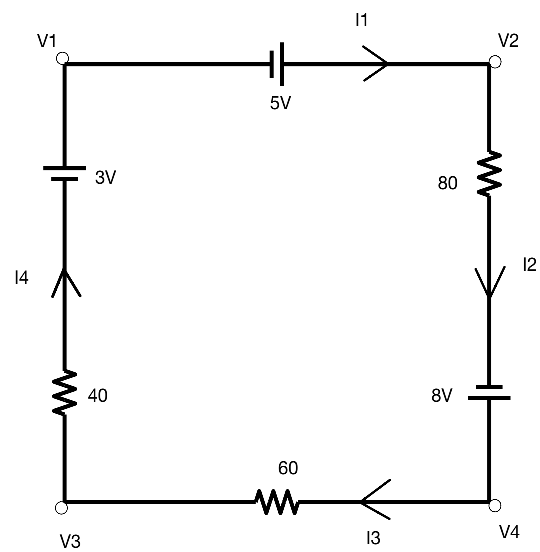
2) (continued) d) (2 points) What number is the matrix C guaranteed to have as an eigenvalue?

e) (4 points) If an associated eigenvector v to the eigenvalue from d) is

$$\begin{bmatrix} 1140/1769 \\ 800/1769 \\ 52873/35380 \\ 1 \end{bmatrix}$$

find the PageRank of P_2.

7) Consider the following electrical circuit (resistance is in Ohms):



a) (8 points) Find the edge-node incidence matrix A.

b) (4 points) Determine the resistance matrix R.

c) (8 points) Set up a matrix equation for finding the currents I_1, I_2, I_3, and I_4 and the potential differences between v_1, v_2, v_3, and v_4.

d) See the next page!

3) d) (continued) (6 points) If the row-reduced matrix you obtain is below, find the currents and potential differences.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4/45 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4/45 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4/45 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4/45 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -53/9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -8/9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -16/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8) Let $v = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$.

a) (3 points) Find a nonzero vector orthogonal to v.

b) (10 points) Find two unit vectors whose angle with v is $21°$.

9) Let

$$A = \begin{bmatrix} 2 & 6 \\ -3 & 5 \\ -9 & -4 \end{bmatrix}.$$

a) (2 points) Find $A^t A$.

b) (4 points) Orthogonally diagonalize $A^t A$, recording all parts of the diagonalization.

c) (4 points) Find the full singular value decomposition of A and record the singular values.

10) (12 points) Let $V = \mathcal{S}$, the space of all infinite sequences of real numbers.

a) (2 points) What are the vectors in V?

b) (10 points) Let

$$W = \{(a_i)_{i=1}^{\infty} \mid \text{there is a } c \in \mathbb{R} \text{ with } a_i = c \text{ for all } i \geq 1\}$$

Show that W is a subspace of V.

11) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y + z \\ -11x + 8y \\ -x - 7y + 5z \end{bmatrix}.$$

a) (9 points) Determine a matrix representation A for T.

b) (6 points) Find a basis for $\ker(T)$.

12) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y + z \\ -11x + 8y \\ -x - 7y + 5z \end{bmatrix}.$$

a) (6 points) Find a basis for $\mathrm{Ran}(T)$.

b) (2 points) Find an orthonormal basis for $\mathrm{Ran}(T)$

c) (6 points) Find a matrix for the orthogonal projection onto $\mathrm{Ran}(T)$.

BONUS 1: (10 points) Let A be an $n \times n$ orthogonal matrix. Show that, for all vectors $x \in \mathbb{R}^n$,

$$\|Ax\|_2 = \|x\|_2.$$

BONUS 2: (10 points) Let $\mathbb{P}[x]$ be the space of all sequences of all polynomials with real coefficients and let $T : \mathbb{P}[x] \to \mathbb{P}[x]$,

$$T\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{i+1}.$$

Find all eigenvalues of T.