Math 227 Final

December 17, 2019

Directions:

1. WRITE YOUR NAME ON THIS TEST!
2. Only calculators are allowed; NO laptops, tablets, or phones.
3. Unless otherwise indicated, decimal approximations for a numerical answer accurate to 4 decimal places are acceptable.
4. If you have a question, raise your hand or come up and ask me.
5. If you had a choice on whether to take this exam, please indicate your understanding of the potential consequences by signing the statement(s) below:

   I understand that by taking this exam, I may lower my grade from what it was before the final.

   Signed: ______________________________

   I had a choice on whether to take this exam, and now that I've seen it, I want to keep my pre-final grade!

   Signed: ______________________________

1. Short Answer. Use the following:

$$A = \begin{bmatrix} 1 & 5 & 0 & 8 & 6 \\ 2 & 10 & -1 & 14 & 12 \\ 3 & 15 & -1 & 22 & 18 \end{bmatrix} \text{ row reduces to } B = \begin{bmatrix} 1 & 5 & 0 & 8 & 6 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (2 points) The homogeneous system $Ax = \vec{0}$ for $\vec{0} \in \mathbb{R}^3$ has

A. No solution B. Infinitely many solutions C. Exactly one solution

D. Either A. or B. E. Either A. or C.

(b) (2 points) For any vector b in $\mathbb{R}^3$, the system $Ax = b$ has

A. No solution B. Infinitely many solutions C. Exactly one solution

D. Either A. or B. E. Either A. or C.

(c) (4 points) Is the matrix B in reduced row echelon form? Briefly explain why or why not.

Recall:

$$A = \begin{bmatrix} 1 & 5 & 0 & 8 & 6 \\ 2 & 10 & -1 & 14 & 12 \\ 3 & 15 & -1 & 22 & 18 \end{bmatrix} \text{ row reduces to } B = \begin{bmatrix} 1 & 5 & 0 & 8 & 6 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(d) (2 points) For matrix A, the column space, Col(A), is a subspace of $\mathbb{R}^m$ where $m =$____________.

(e) (2 points) For matrix A, the row space, Row(A), is a subspace of $\mathbb{R}^m$ where $m =$____________.

(f) (2 points) For matrix A, the null space, ker(A), is a subspace of $\mathbb{R}^m$ where $m =$____________.

(g) (6 points) Referring to A as above, for each of the spaces Col(A), Row(A), and ker(A), write down a nonzero vector in the space.

2. Mark each item as True or False. No justification is necessary.

   (a) (2 points) True or False: For every matrix A, the homogeneous system $A\vec{x} = \vec{0}$ has at least one solution.

   (b) (2 points) True or False: If A is a 5×11 matrix, then the system $A\vec{x} = \vec{b}$ has at least one solution for all $\vec{b} \in \mathbb{R}^5$.

   (c) (2 points) True or False: If A is an invertible $n \times n$ matrix, then the system $A\vec{x} = \vec{b}$ has at least one solution for all $\vec{b} \in \mathbb{R}^n$.

   (d) (2 points) True or False: If A is an $n \times n$ matrix with $\det(A) = 0$, then A^{-1} exists.

   (e) (2 points) True or False: If A is an $n \times n$ diagonal matrix, then A^{-1} exists.

3. Determine whether the following computations can be done. If the computation cannot be done, briefly explain why. If the computation can be done, give the dimensions of the resulting output.

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) (2 points) $A\vec{v}$

(b) (2 points) $\vec{v}A$

(c) (2 points) AB

(d) (2 points) AC

(e) (2 points) $C^T + B$

4. (12 points) Solve the following system of equations BY HAND, using any manner at your disposal, or show there is no solution. SHOW YOUR WORK.

$$\begin{aligned} x - 2y + 3z &= 1 \\ x - y + 3z &= 2 \\ 2x - 3y + 7z &= 2 \end{aligned}$$

5. Let

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}.$$

a) (10 points) Compute all eigenvalues of A BY HAND.

b) (10 points) Show that $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors of A.

6. Find the matrix of the linear transformations on $\mathbb{R}^3$ that, in homogeneous coordinates,

   a) (3 points) scales a 2-vector down by a factor of 6.

   b) (4 points) shifts a 2-vector up 5 units and left 14 units

   c) (5 points) rotates a 2-vector by $\pi/6$ radians counterclockwise.

   d) (2 points) If A, B, and C are the matrices from parts a), b), and c), respectively, in what order do you write the product of A, B, and C if you first scale, then shift, then rotate?

7. Given the points $(0, 2)$, $(-3, 4)$, $(5, 8)$ and $(9, -16)$ in $\mathbb{R}^2$, find the best-fit line to the points by

   a) (6 points) Finding a system of linear equations that represents a "solution" to the problem,

   b) (3 points) Writing the problem as a matrix equation $Ax = b$,

   c) (8 points) Finding the system $A^tAx = A^tb$, computing both A^tA and A^tb,

   d) (7 points) Solving the system in c) and producing the polynomial.

8. Given the simplified link diagram between webpages P_1, P_2 and P_3 described by

- P_1 links to P_3 and P_2
- P_2 links to P_1 and P_3
- P_3 links to P_2

a) (5 points) Construct the link matrix A.

b) (3 points) Find the normalized matrix B.

c) (5 points) Write down a formula for the PageRank matrix C, using $d = .85 = 17/20$, but DO NOT COMPUTE C.

d) (4 points) If an associated eigenvector v to the eigenvalue $\lambda = 1$ is

$$\begin{bmatrix} 40/57 \\ 74/57 \\ 1 \end{bmatrix}$$

find the PageRank of P_1.

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x - 7y \\ -13x + 42y \\ 101x + 8y \end{bmatrix}.$$

a) (9 points) Find a matrix representation A for T.

b) (4 points) Write down a formula for the orthogonal projection onto $\mathrm{Ran}(T)$.

10. Let

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}.$$

Denote by W the subspace of $\mathbb{R}^3$ consisting of all linear combinations of v_1, v_2, and v_3.

(a) (7 points) Determine whether v_3 is a linear combination of v_1 and v_2.

(b) (7 points) Find a nonzero vector orthogonal to every vector in W.

(c) (7 points) Determine the orthogonal projection onto $W^\perp$.

11. (13 points) Let

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + 2z = 0 \right\}$$

Show that W is a subspace of $\mathbb{R}^3$.

12. (10 points) Let

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + y \geq 0 \right\}$$

Show that S is not a subspace of $\mathbb{R}^2$.

13. (15 points) Define $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$,

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Show that ± 1 are the only eigenvalues of T.