Math 227 Final

December 14, 2021

Directions:

1. WRITE YOUR NAME ON THIS TEST!
2. Only calculators are allowed; NO laptops, tablets, or phones.
3. Unless otherwise indicated, decimal approximations for a numerical answer accurate to 4 decimal places are acceptable.
4. If you have a question, raise your hand or come up and ask me.
5. If you had a choice on whether to take this exam, please indicate your understanding of the potential consequences by signing the statement(s) below:

   I understand that by taking this exam, I may lower my grade from what it was before the final.

   Signed: ______________________________

   I had a choice on whether to take this exam, and now that I've seen it, I want to keep my pre-final grade!

   Signed: ______________________________

1. a) (4 points) Is it possible for a system of linear equations to have exactly 42 solutions? Why or why not?

   b) (3 points) If V is a vector space, what are the two operations on V, i.e., what makes a vector space?

   c) (3 points) What are the possible geometric descriptions for subspaces of $\mathbb{R}^2$?

   d) (4 points) What are the possible geometric descriptions for subspaces of $\mathbb{R}^3$?

   e) (2 points) Fill in the blank: Every linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by a ____________.

2. (CHANGE NUMBERS) Determine whether the following computations can be done. If the computation cannot be done, simply state it cannot be done. If the computation can be done, give the dimensions of the resulting output.

$$A = \begin{bmatrix} 1 & 4 & -2 & 6 \\ 2 & 0 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 2 \\ 8 & 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

(a) (2 points) $A \cdot \vec{v}$

(b) (2 points) $\vec{v} \cdot A$

(c) (2 points) $A \cdot B$

(d) (2 points) $B \cdot A$

(e) (2 points) $A^T + B$

3. Find a ***QUADRATIC*** interpolating polynomial through the points $(4, -3)$, $(2, 3)$ and $(-2, 2)$ by

   a) (10 points) writing down a system of linear equations that determines the coefficients of the polynomial, then

   b) (15 points) solving the resulting system of equations BY HAND, using any manner at your disposal and SHOWING YOUR WORK, and finally

   c) (4 points) writing down the polynomial.

4. Find a single 3×3 matrix that, in homogeneous coordinates,

a) (4 points) shifts a 2-vector right 7 units and down 8 units,

b) (4 points) rotates a 2-vector by $7\pi/6$ radians counterclockwise,

c) (3 points) scales the x-coordinate of a 2-vector down by a factor of 5 and scales the y-coordinate up by a factor of 10.

d) (2 points) If A, B, and C are the matrices from parts a), b), and c), respectively, in what order do you write the product of A, B, and C if you first scale, then shift, then rotate?

5. Given the points $(-7,-2)$, $(5,-7)$, and $(-9,-5)$ in $\mathbb{R}^2$, find the best-fit line to the points by

   a) (6 points) Finding a system of linear equations that represents a "solution" to the problem,

   b) (3 points) Writing the problem as a matrix equation $A \cdot \vec{x} = \vec{b}$,

   c) (6 points) Finding the system $A^t \cdot A \cdot \vec{x} = A^t \cdot \vec{b}$, computing both $A^t \cdot A$ and $A^t \cdot \vec{b}$,

   d) (7 points) Solving the system in c) and producing the polynomial.

6. Let

$$A = \begin{bmatrix} 2 & -2 & -5 \\ 0 & 5 & 0 \\ -1 & 0 & -2 \end{bmatrix}.$$

a) (12 points) Compute all eigenvalues of A BY HAND.

b) (9 points) Check that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ -8 \\ -1 \end{bmatrix}$ are eigenvectors corresponding to the eigenvalues you found in a) BY HAND.

7. Given the simplified link diagram between webpages P_1, P_2, $P3$, and P_4 described by

- P_1 links to P_2, P_3 and P_4
- P_2 links to P_3 and P_4
- P_3 links to P_2 and P_4
- P_4 links to P_1

a) (5 points) Construct the link matrix A.

b) (3 points) Find the normalized matrix B.

c) (8 points) Find the PageRank matrix C, using $d = 17/20$.

d) (4 points) If an associated eigenvector v to the eigenvalue $\lambda = 1$ is

$$\begin{bmatrix} 4287 \\ 3080 \\ 3080 \\ 4389 \end{bmatrix}$$

find the PageRank of P_4.

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y - x \\ -4y - 2x \\ x + 3y \end{bmatrix}.$$

a) (9 points) Find a matrix representation A for T.

b) (6 points) Find three nonzero, nonparallel vectors in $\mathrm{Ran}(T)$, then describe $\mathrm{Ran}(T)$ geometrically. You do not have to show your work.

b) (4 points) Write down a formula for the orthogonal projection onto $\mathrm{Ran}(T)$. DO NOT ACTUALLY COMPUTE THE PROJECTION.

9. Let

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}.$$

Let $W = \text{span}(v_1, v_2, v_3)$.

(a) (7 points) Determine whether v_3 is in the span of v_1 and v_2.

(b) (7 points) Find an orthonormal basis for W.

(c) (5 points) Find a nonzero vector $\vec{x}$ that is orthogonal to every vector in W.

(d) (4 points) Determine the orthogonal projection onto $\text{span}(x)$.

10. a) (6 points) Suppose A is a 4×4 matrix and there is a *nonzero* vector $\vec{v} \in \mathbb{R}^4$ with $A\vec{v} = \vec{0}$. Is A the zero matrix? Why or why not?

    b) (6 points) Suppose A is an invertible 4×4 matrix. What number do you definitely know is NOT an eigenvalue for A? Why?

11. (15 points) Let $\mathcal{S}(\mathbb{R})$ denote all sequences of real numbers and let

$$W = \{(a_n)_{n=1}^{\infty} \mid a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3\}.$$

Show that W is a subspace of $\mathcal{S}(\mathbb{R})$.