Math 227 Final

December 11, 2023

Directions:

1. WRITE YOUR NAME ON THIS TEST!
2. Only calculators are allowed; NO laptops, tablets, or phones.
3. Unless otherwise indicated, decimal approximations for a numerical answer accurate to 4 decimal places are acceptable.
4. If you have a question, raise your hand or come up and ask me.
5. If you had a choice on whether to take this exam, please indicate your understanding of the potential consequences by signing the statement(s) below:

   I understand that by taking this exam, I may lower my grade from what it was before the final.

   Signed: ______________________________

   I had a choice on whether to take this exam, and now that I've seen it, I want to keep my pre-final grade!

   Signed: ______________________________

1. a) Is it possible for a system of linear equations to have exactly 116 solutions? Why or why not?

   b) If V is a vector space, what are the two operations on V, i.e., what makes a vector space?

   c) What are the possible geometric descriptions for subspaces of $\mathbb{R}^2$?

   d) What are the possible geometric descriptions for subspaces of $\mathbb{R}^3$?

   e) Fill in the blank: Every linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by a ____________.

2. Determine whether the following computations can be done. If the computation cannot be done, simply state it cannot be done. If the computation can be done, give the dimensions of the resulting output.

$$A = \begin{bmatrix} 5 & -2 & 0 & 17 \\ 2 & 1 & 42 & -16 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 7 \\ -4 & 3 \\ 11 & 1 \\ 13 & -8 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 8 \\ 1 \\ 4 \\ -6 \end{bmatrix}.$$

(a) $A \cdot \vec{v}$

(b) $B \cdot \vec{v}$

(c) $A \cdot B^t$

(d) $B \cdot A^t$

(e) $A + B$

3. Find a $\boldsymbol{CUBIC}$ interpolating polynomial through the points $(1,4)$, $(-1,8)$, $(-2,7)$ and $(0,5)$ by

   a) writing down a system of linear equations that determines the coefficients of the polynomial, then

   b) solving the resulting system of equations BY HAND, using any manner at your disposal and SHOWING YOUR WORK, and finally

   c) writing down the polynomial.

4. Find a single 3×3 matrix that, in homogeneous coordinates,

   a) scales the x-coordinate of a 2-vector up by a factor of 11 and scales the y-coordinate up by a factor of 8,

   b) shifts a 2-vector right 16 units and up 9 units,

   c) rotates a 2-vector by $\pi/6$ radians counterclockwise.

   d) If A, B, and C are the matrices from parts a), b), and c), respectively, in what order do you write the product of A, B, and C if you first rotate, then shift, then scale?

5. Given the points $(1,4)$, $(-1,8)$, $(-2,7)$ and $(0,5)$ in $\mathbb{R}^2$, find the best-fit ***LINE*** to the points by

   a) Finding a system of linear equations that represents a "solution" to the problem,

   b) Writing the problem as a matrix equation $A \cdot \vec{x} = \vec{b}$,

   c) Finding the system $A^t \cdot A \cdot \vec{x} = A^t \cdot \vec{b}$, computing both $A^t \cdot A$ and $A^t \cdot \vec{b}$,

   d) Solving the system in c) and producing the polynomial.

6. Let

$$A = \begin{bmatrix} 8 & 1 & -4 \\ 0 & -2 & 0 \\ 5 & 0 & -1 \end{bmatrix}.$$

a) What is the one vector in $\mathbb{R}^3$ that has no possibility of being an eigenvector for A?

b) Compute all eigenvalues of A BY HAND.

c) If $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for A, find two other eigenvectors for A.

7. Given the simplified link diagram between webpages P_1, P_2, P_3, and P_4 described by

- P_1 links to P_2 and P_4,
- P_2 doesn't link to anything,
- P_3 links to P_1, P_2, and P_4,
- P_4 links to P_2

a) Construct the link matrix A.

b) Find the normalized matrix B.

c) Find the PageRank matrix C, using $d = 17/20$.

d) If an associated eigenvector $\vec{v}$ to the eigenvalue $\lambda = 1$ is

$$\begin{bmatrix} 61600 \\ 162393 \\ 48000 \\ 87780 \end{bmatrix}$$

find the PageRank of P_1.

8. a) Find the area of the parallelogram with vertices $(0,0)$, $(6,1)$, $(-2,4)$, and $(4,5)$.

b) Find a nonzero vector of length 7 that is orthogonal to both $\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 12 \\ -6 \end{bmatrix}$

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y - z \\ x - y \\ x + y - 2z \end{bmatrix}.$$

a) Determine a matrix representation A for T.

b) Describe the range of T geometrically, with reasoning and (potentially) calculations to support your answer.

c) Find the kernel of T.

d) Calculate a matrix for the orthogonal projection onto the kernel of T.

10. Let

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -3 \\ 8 \\ 48 \end{bmatrix}.$$

Let $W = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

(a) Determine whether $\vec{v}_1$ is in the span of $\vec{v}_2$ and $\vec{v}_3$.

(b) Find an orthonormal basis for W.

11. a) Give an example of a nonzero 2×2 matrix A and a nonzero vector $\vec{v}$ with $A\vec{v} = \vec{0}$.

b) Show that if A is an $n \times n$ matrix and $A^t A$ is invertible, then A is invertible.

12. Let $\mathcal{F}(\mathbb{R})$ denote the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, and let

$$W = \left\{ f \in \mathcal{F}(\mathbb{R}) \;:\; f(x) = f(-x) \text{ for all } x \in \mathbb{R} \right\}.$$

That is, W consists of all *even* function from $\mathbb{R}$ to $\mathbb{R}$.

a) Write down three functions in W.

b) Write down a function that is NOT in W (if possible).

c) Show that W is a subspace of $\mathcal{F}(\mathbb{R})$.