Name:

Math 227 Final

April 23, 2018

Directions:

1. WRITE YOUR NAME ON THIS TEST!
2. Wolfram Alpha or a similar program may be used for all computational problems.
3. Unless otherwise indicated, decimal approximations for a numerical answer accurate to 4 decimal places are acceptable.
4. If you have a question, raise your hand or come up and ask me.
5. If you had a choice on whether to take this exam, please indicate your understanding of the potential consequences by signing the statement below:

I understand that by taking this exam, I may lower my grade from what it was before the final.

Signed: ____________________________

1) (12 points) Solve the following system of equations BY HAND, using any manner at your disposal, or show there is no solution. SHOW YOUR WORK.

$$4x + y - 3z = 11$$
$$2x + 3y - z = 1$$
$$3x - 2y + 5z = 21$$

2) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 2 & -1 \end{bmatrix}.$$

a) (10 points) Compute all eigenvalues of A BY HAND.

b) (3 points) Find an associated eigenvector for each eigenvalue from a).

c) (9 points) Check that each actually vector from b) actually is an eigenvector BY HAND.

3) Find the matrix of the linear transformations on $\mathbb{R}^3$ that, in homogeneous coordinates,

a) (3 points) scales a 2-vector down by a factor of 4.

b) (4 points) shifts a 2-vector down 10 units and right 7 units

c) (5 points) rotates a 2-vector by $3\pi/4$ radians counterclockwise

d) (6 points) rotates a 2-vector by $3\pi/4$ radians counterclockwise, then scales a 2-vector down by a factor of 4, and finally shifts a 2-vector down 10 units and right 7 units.

4) Find the interpolating cubic through the points $(1, 4)$, $(-3, 6)$, $(2, 2)$ and $(-1, 7)$ in $\mathbb{R}^2$ by,

a) (10 points) writing down a system of linear equations that determines the coefficients of the polynomial, then

b) (6 points) finding the solution to the equation and writing down the polynomial.

5) Given the points $(1, 4)$, $(-3, 6)$, $(2, 2)$ and $(-1, 7)$ in $\mathbb{R}^2$, find the best-fit quadratic to the points by

a) (8 points) Finding a system of linear equations that represents a "solution" to the problem,

b) (6 points) Writing the problem as a matrix equation $Ax = b$,

c) (5 points) Finding the system $A^t Ax = A^t b$, computing both $A^t A$ and $A^t b$,

d) (5 points) Solving the system in c) and producing the polynomial.

6) Given the simplified link diagram between webpages P_1, P_2, P_3, and P_4 described by

- P_1 links to P_2 and P_3
- P_2 links to P_1 and P_4
- P_3 links to P_1, P_2, and P_4
- P_4 doesn't link to anything

a) (8 points) Construct the link matrix A.

b) (5 points) Find the normalized matrix B.

c) (8 points) Calculate the PageRank matrix C, using $d = .85 = 17/20$.

d) (6 points) Find the associated eigenvector v with all positive entries whose 1-norm is equal to one and find the PageRank of P_3.

7) Let

$$A = \begin{bmatrix} 38 & -8 \\ -7 & -26 \\ -7 & -26 \end{bmatrix}.$$

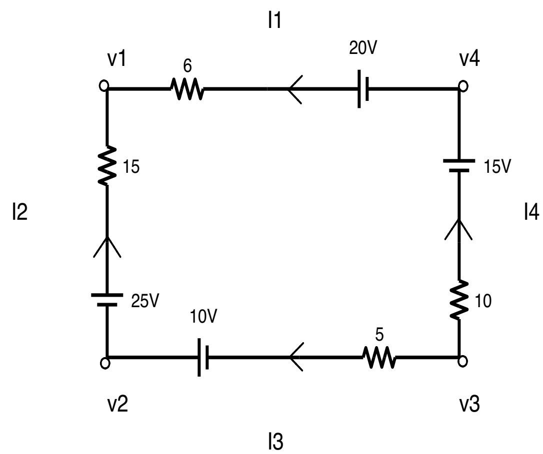
a) (2 points) Find A^tA.

b) (4 points) Orthogonally diagonalize A^tA and record the eigenvalues.

c) (4 points) Find the full singular value decomposition of A and record the singular values.

d) (3 points) Find the score matrix T and record the first principal component of each matrix.

8) Consider the following electrical circuit (resistance is in Ohms):



a) (8 points) Find the edge-node incidence matrix A.

b) (4 points) Determine the resistance matrix R.

c) (8 points) Set up a matrix equation for finding the currents I_1, I_2, I_3, and I_4 and the potential differences between v_1, v_2, v_3, and v_4.

d) (8 points) Find the currents I_1, I_2, I_3 and I_4 and potential differences between v_1, v_2, v_3 and v_4.

9) (6 points) Let

$$v_1 = \begin{bmatrix} 6 \\ 42 \\ -10 \end{bmatrix}, \; v_2 = \begin{bmatrix} 8 \\ -13 \\ -1 \end{bmatrix}$$

Find two different vectors v and w in $span\{v_1, v_2\}$ that are neither a scalar multiple of v_1 nor a scalar multiple of v_2. Then find a vector u that is NOT in $span\{v_1, v_2\}$. For the last part, be sure to show that your answer is correct.

10) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$,

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 5x + 8y - 10z \\ 4z + 9w \\ 13x - 5y + 9z - w \end{bmatrix}.$$

a) (9 points) Determine a matrix representation A for T.

b) (5 points) Find an orthonormal basis for $\operatorname{ran}(T)$.

11) (10 points) If $A = \begin{bmatrix} 5 & 8 & -10 \\ 0 & 0 & 4 \\ 13 & -5 & 9 \end{bmatrix}$, let

$$W = \{B \in M_3(\mathbb{R}) \mid A^t B A = B\}.$$

Show that W is a subspace of $M_3(\mathbb{R})$.

12) (10 points) Let V be an arbitrary 2-dimensional subspace of $\mathbb{R}^2$. Find an orthonormal basis for W.

BONUS 1: (10 points) Let A be an $m \times n$ matrix. Show that, for all vectors $y \in \text{ran}(A)$ and $z \in \ker(A^t)$, $y \cdot z = 0$.

BONUS 2: (10 points) Let S be the space of all sequences of real numbers and define $T : S \to S$,

$$T((a_n)_{n=1}^{\infty}) = (a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots).$$

In compact form, $T((a_n)_{n=1}^{\infty}) = \left(\sum_{k=1}^{n} a_k \right)_{n=1}^{\infty}$. Find all eigenvalues of T.