1) a)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = A$$

b) $$A' = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1/2 & 1/3 \\ 0 & 0 & 1/3 \\ 1 & 1/2 & 1/3 \end{bmatrix}$$

c) $$C = \frac{17}{20} B + \frac{1}{20} E$$

$$C = \begin{bmatrix} 1/20 & 19/40 & 1/3 \\ 1/20 & 1/20 & 1/3 \\ 9/10 & 19/40 & 1/3 \end{bmatrix}$$

d) $0 = \det(C - \lambda I_3)$

$$= \det \begin{bmatrix} 1/20 - \lambda & 19/40 & 1/3 \\ 1/20 & 1/20 - \lambda & 1/3 \\ 9/10 & 19/40 & 1/3 - \lambda \end{bmatrix}$$

$$= \begin{matrix} 1/20 - \lambda & 19/40 & 1/3 & 1/20 - \lambda & 19/40 \\ 1/20 & 1/20 - \lambda & 1/3 & 1/20 & 1/20 - \lambda \\ 9/10 & 19/40 & 1/3 - \lambda & 9/10 & 19/40 \end{matrix}$$

$$= (1/20 - \lambda)^2 (1/3 - \lambda) + 19/40 \cdot 3/10$$
$$+ 1/20 \cdot 19/40 - 1/3 (1/20 - \lambda) 9/10$$
$$- (1/20 - \lambda) \cdot 1/3 \cdot 19/40 - 19/40 \cdot 1/20 (1/3 - \lambda)$$

$$= (1/20 - \lambda)^2 (1/3 - \lambda) + \frac{57}{400} + \frac{19}{2400}$$
$$- \frac{3}{10}(1/20 - \lambda) - \frac{19}{120}(1/20 - \lambda)$$
$$- 19/800 (1/3 - \lambda)$$

$$0 = \frac{289}{2400} + \left(-\lambda^3 + \frac{13}{30}\lambda^2 - \frac{43\lambda}{1200}\right) + \frac{3\lambda}{10}$$

$$+ \frac{19\lambda}{120} + \frac{19\lambda}{800}$$

$$0 = \frac{289}{2400} + \frac{357}{800}\lambda + \frac{13}{30}\lambda^2 - \lambda^3$$

$$0 = (1-\lambda)\left(\lambda^2 + a\lambda + \frac{289}{2400}\right)$$

$$\lambda^2 - a\lambda^2 = \frac{13}{30}$$

$$1 - a = \frac{13}{30}, \quad a = \frac{17}{30}$$

$$0 = (1-\lambda)\left(\lambda^2 + \frac{17}{30}\lambda + \frac{289}{2400}\right)$$

$$\lambda = 1, \quad \lambda = \frac{-\frac{17}{30} \pm \sqrt{\left(\frac{17}{30}\right)^2 - 4\left(\frac{289}{2400}\right)}}{2}$$

$$\lambda = -\frac{17}{60} \pm \frac{\sqrt{\frac{289}{900} - \frac{289}{600}}}{2}$$

$$= -\frac{17}{60} \pm \frac{17}{20}\sqrt{\frac{1}{9} - \frac{1}{6}}$$

$$= -\frac{17}{60} \pm \frac{17}{20}\sqrt{-\frac{2}{36}}$$

$$= -\frac{17}{60} \pm \frac{17}{120}\sqrt{2}\, i$$

e) $\|v\|_1 = 1 + \frac{800}{2109} + \frac{20}{37}$

$$= \frac{4049}{2109}$$

PageRank of $P_2 = \frac{2109}{4049} \cdot \frac{800}{2109} = \frac{800}{4049}$

$\approx .1976$

2) a) $-m+b=2$

$b=8$

$3m+b=-1$

b) $\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -1 \end{bmatrix}$

c) $A^t A = \begin{bmatrix} -1 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 10 & 2 \\ 2 & 3 \end{bmatrix}$

$A^t b = \begin{bmatrix} -1 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 9 \end{bmatrix}$

d) $\begin{bmatrix} 10 & 2 & -5 \\ 2 & 3 & 9 \end{bmatrix}$

$R1 \leftrightarrow R2 \quad \begin{bmatrix} 2 & 3 & 9 \\ 10 & 2 & -5 \end{bmatrix}$

$-5R1+R2 \quad \begin{bmatrix} 2 & 3 & 9 \\ 0 & -13 & -50 \end{bmatrix}$

$R2/13 \quad \begin{bmatrix} 2 & 3 & 9 \\ 0 & 1 & 50/13 \end{bmatrix}$

$R1/2 \quad \begin{bmatrix} 1 & 3/2 & 9/2 \\ 0 & 1 & 50/13 \end{bmatrix}$

$-3/2\,R2+R1 \quad \begin{bmatrix} 1 & 0 & -33/26 \\ 0 & 1 & 50/13 \end{bmatrix} \quad y = \frac{-33}{26}x + \frac{50}{13}$

3) a) $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\3\end{bmatrix}$

$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$A = \begin{bmatrix}-1 & 1\\0 & 1\\3 & 1\end{bmatrix}$

b) Basis $\left\{ \begin{bmatrix}-1\\0\\3\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} = \mathcal{B}$

These two vectors are not scaler multiples, so they are linearly independent. The column space is all vectors of the form $x\begin{bmatrix}-1\\0\\3\end{bmatrix} + y\begin{bmatrix}1\\1\\1\end{bmatrix}$, so this shows $\mathcal{B}$ is a basis

c) Gram-Schmidt

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \quad \|v_1\|_2 = \sqrt{10}$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}}{10} \cdot \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 6/5 \\ 1 \\ 2/5 \end{bmatrix} = 1/5 \begin{bmatrix} 6 \\ 5 \\ 2 \end{bmatrix}$$

$$\|v_2\|_2 = \frac{1}{5} \sqrt{65}$$

Orthonormal basis $\{w_1, w_2\}$

where $w_1 = \frac{v_1}{\|v_1\|_2} = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$

and $w_2 = \frac{v_2}{\|v_2\|_2} = \frac{1}{\sqrt{65}} \begin{bmatrix} 6 \\ 5 \\ 2 \end{bmatrix}$

4) i) Zero vector: $(0,0,0,\ldots)$

$\lim_{n\to\infty} 0 = 0$, so the zero vector is in W.

ii) Suppose $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty} \in W$ and $c \in \mathbb{R}$. Then $\exists\, a, b$

$\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$.

Then since $c(a_n)_{n=1}^{\infty} + (b_n)_{n=1}^{\infty} = (ca_n + b_n)_{n=1}^{\infty}$,

$\lim_{n\to\infty} (ca_n + b_n)$

$= \lim_{n\to\infty} (ca_n) + \lim_{n\to\infty} b_n$

$= c\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

$= ca + b$ ✓

5) Since W is a subspace, choose an orthonormal basis $\{w_1, w_2, \dots, w_k\}$ for W where $k = \dim(W)$. Then

$$P = \sum_{j=1}^{k} w_j w_j^t$$

is the orthogonal projection from $\mathbb{R}^n$ onto W. Consider $T = I_n - P$.

Then for all $x \in W$,

$$Tx = (I_n - P)x = x - Px$$
$$= x - x = 0.$$

Now suppose $Tx = 0$ for some $x \in \mathbb{R}^n$.

We can write $x = Px + (I-P)x$,

so that

$$0 = Tx = T(Px + (I-P)x)$$

$$= T(Px) + T((I-P)x)$$

$$= 0 + (I-P)x$$

So $(I-P)x = 0$ and $x = Px$

$\Rightarrow x \in W$.

6) Since N is a nilpotent matrix, the only eigenvalues of N are zero. This implies that the characteristic polynomial of N is $p(x) = x^n$. By the Cayley-Hamilton theorem,

$p(N) = 0$, so $N^n = 0$.

Then $k \leq n$ since k is the smallest whole number with $N^k = 0$.