# Math 227 Final

April 26, 2022

**Directions:**

1. WRITE YOUR NAME ON THIS TEST!
2. Only calculators are allowed; NO laptops, tablets, or phones.
3. Unless otherwise indicated, decimal approximations for a numerical answer accurate to 4 decimal places are acceptable.
4. If you have a question, raise your hand or come up and ask me.
5. If you had a choice on whether to take this exam, please indicate your understanding of the potential consequences by signing the statement(s) below:

   I understand that by taking this exam, I may lower my grade from what it was before the final.

   **Signed:** ______________________________

   I had a choice on whether to take this exam, and now that I've seen it, I want to keep my pre-final grade!

   **Signed:** ______________________________

1. a) Is it possible for a system of linear equations to have exactly 116 solutions? Why or why not?

   b) If $V$ is a vector space, what are the two operations on $V$, i.e., what makes a vector space?

   c) What are the possible geometric descriptions for subspaces of $\mathbb{R}^2$?

   d) What are the possible geometric descriptions for subspaces of $\mathbb{R}^3$?

   e) Fill in the blank: Every linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by a ____________.

2. Determine whether the following computations can be done. If the computation cannot be done, simply state it cannot be done. If the computation can be done, give the dimensions of the resulting output.

$$A = \begin{bmatrix} 1 & 4 & -2 & 6 \\ 2 & 0 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 2 \\ 8 & 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

(a) $B \cdot \vec{v}$

(b) $\vec{v} \cdot B$

(c) $A \cdot B$

(d) $B^t \cdot \vec{v}$

(e) $A^T + B^T$

3. In this problem, you will find an interpolating polynomial of the form

$$p(x) = ax^n + bx^{n-1} + \dots \tag{$\star$}$$

which passes exactly through the following points:

$$(x_1, y_1) = (0, -4), \quad (x_2, y_2) = (2, 10), \quad (x_3, y_3) = (-2, 6), \quad \text{and} \quad (x_4, y_4) = (-1, 4).$$

**<u>Note</u>: If you are unsuccessful in answering part a) and/or part b) of this problem, please proceed to part c), which can be solved independently for full credit.**

a) What is the *lowest* degree polynomial which you would expect to pass exactly through the above points?

b) For polynomial interpolation of the above data set, we require that

$$p(x_1) = y_1, \quad p(x_2) = y_2, \quad p(x_3) = y_3, \quad \text{and} \quad p(x_4) = y_4,$$

where $p(x)$ is given by equation ($\star$) above. Applying these interpolation conditions, write out a linear system of equations which relates the unknown coefficients of the interpolating polynomial to the given data values.

c) Solve the system from part b) and use your solution to write down the interpolating polynomial.

*Make sure to show all steps clearly and label all row operations utilized in your solution.*

If you were unsuccessful in formulating a linear system in part (b), solve the following linear system instead:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ -8 & 4 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ -27 & 9 & -3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -6 \\ -10 \end{pmatrix}.$$

4. Find a single $3 \times 3$ matrix that, in homogeneous coordinates,

a) scales the $x$-coordinate of a 2-vector up by a factor of 3 and scales the $y$-coordinate up by a factor of 8,

b) rotates a 2-vector by $3\pi/4$ radians counterclockwise,

c) shifts a 2-vector left 10 units and down 12 units.

d) If $A$, $B$, and $C$ are the matrices from parts a), b), and c), respectively, in what order do you write the product of $A$, $B$, and $C$ if you first scale, then shift, then rotate?

5. Given the points $(-1,6)$, $(4,-7)$, $(2,5)$, $(-3,-2)$, and $(1,13)$ in $\mathbb{R}^2$, find the best-fit ***LINE*** to the points by

   a) Finding a system of linear equations that represents a "solution" to the problem,

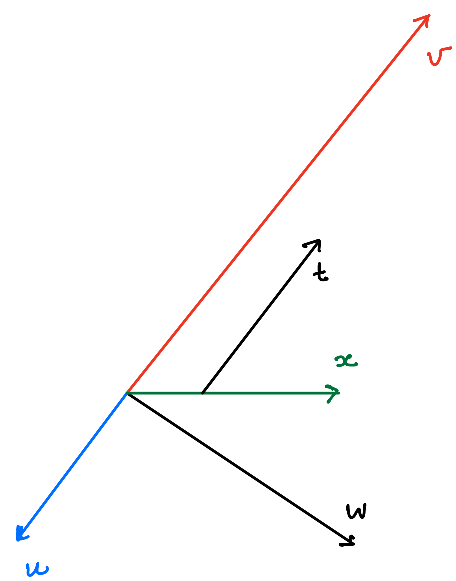
   b) Writing the problem as a matrix equation $A \cdot \vec{x} = \vec{b}$,

   c) Finding the system $A^t \cdot A \cdot \vec{x} = A^t \cdot \vec{b}$, computing both $A^t \cdot A$ and $A^t \cdot \vec{b}$,

   d) Solving the system in c) and producing the polynomial.

6. a) Is $\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$? If so, what is the associated eigenvalue? Justify your answer.

b) If the vector $\vec{v}$ (shown below) is an eigenvector of a matrix $B$ which of the following vectors ($\vec{v}, \vec{t}, \vec{x}, \vec{w}$, and $\vec{u}$) can be the result of the product $B\vec{v}$? Carefully justify your answer.



7. Given the simplified link diagram between webpages $P_1, P_2$, $P_3$, and $P_4$ described by

   - $P_1$ links to $P_4$,
   - $P_2$ links to $P_1$ and $P_4$,
   - $P_3$ links to $P_2$ and $P_4$,
   - $P_4$ links to $P_1$, $P_2$, and $P_3$,

   a) Construct the link matrix $A$.

   b) Find the normalized matrix $B$.

   c) Find the PageRank matrix $C$, using $d = 17/20$.

   d) If an associated eigenvector $v$ to the eigenvalue $\lambda = 1$ is

$$\begin{bmatrix} 197813 \\ 175560 \\ 123200 \\ 325959 \end{bmatrix}$$

   find the PageRank of $P_2$.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 4y - 5z \\ 2x + 6z \\ 5y - 10z \end{bmatrix}.$$

a) Find the range of $T$.

b) Describe the range of $T$ geometrically, with reasoning and (potentially) calculations to support your answer.

9. Let

$$\vec{v}_1 = \begin{bmatrix} 10 \\ 0 \\ -9 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 8 \\ 1 \\ 6 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 6 \\ 2 \\ 21 \end{bmatrix}.$$

Let $W = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

(a) Determine whether $\vec{v}_2$ is in the span of $\vec{v}_1$ and $\vec{v}_3$.

(b) Find an orthonormal basis for $W$.

(c) Determine the orthogonal projection onto $W$.

10. a) Suppose $A$ is a $6 \times 6$ matrix and there is a *nonzero* vector $\vec{v} \in \mathbb{R}^6$ with $A^t \vec{v} = \vec{0}$. Is $A$ the zero matrix? Why or why not?

    b) Suppose $A$ is an invertible $6 \times 6$ matrix. What number do you definitely know is NOT an eigenvalue for $A^{-1}$? Why?

11. Let $\mathcal{F}(\mathbb{R})$ denote all functions from $\mathbb{R} \to \mathbb{R}$ and let

$$W = \{f \in \mathcal{F}(\mathbb{R}) \mid \lim_{x \to 42} f(x) \text{ exists}\}.$$

a) Write down three functions in $W$.

b) Write down a function that is NOT in $W$ (if possible).

c) Show that $W$ is a subspace of $\mathcal{F}(\mathbb{R})$.

12. Let $\mathcal{S}$ be the following subset of $\mathbb{R}^3$:

$$\mathcal{S} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x^2 + y^2 + z^2 \leq 1 \right\}.$$

Show that $\mathcal{S}$ is NOT a subspace of $\mathbb{R}^3$.