Winter '22 final

1 a) No - only zero, one, or infinitely many solutions

b) vector addition and scalar multiplication

c) point, line, plane ($\mathbb{R}^2$)

d) point, line, plane, $\mathbb{R}^3$

e) matrix

2) a) No

b) No

c) yes, 2×2

d) yes, 2×1

e) No

3) a) 3

b) $y = ax^3 + bx^2 + cx + d$

$-4 = d$

$10 = 8a + 4b + 2c + d$

$6 = -8a + 4b - 2c + d$

$4 = -a + b - c + d$

c) $d = -4$, so substituting,

$14 = 8a + 4b + 2c$

$10 = -8a + 4b - 2c$

$8 = -a + b - c$

$14 = 8a + 4b + 2c$

$+ (10 = -8a + 4b - 2c)$

$$24 = 8b$$

$$b = 3$$

Substitute in last 2 equations

$$-2 = -8a - 2c$$

$$5 = -a - c$$

$$-2 = -8a - 2c$$

$$+ \quad -2(5 = -a - c)$$

$$-12 = -6a$$

$$a = 2$$

$$5 = -a - c$$

$$c = 7$$

$$y = 2x^3 + 3x^2 - 7x - 4$$

4) a)

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} \cos(3\pi/4) & -\sin(3\pi/4) & 0 \\ \sin(3\pi/4) & \cos(3\pi/4) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c)

$$\begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & -12 \\ 0 & 0 & 1 \end{bmatrix}$$

d) $B \cdot C \cdot A$

5) a) $y = mx + b$

$6 = -m + b$

$-7 = 4m + b$

$5 = 2m + b$

$-2 = -3m + b$

$13 = m + b$

b) $\begin{bmatrix} -1 & 1 \\ 4 & 1 \\ 2 & 1 \\ -3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 5 \\ -2 \\ 13 \end{bmatrix}$

c) $A^t \cdot A = \begin{bmatrix} -1 & 4 & 2 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 4 & 1 \\ 2 & 1 \\ -3 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 31 & 3 \\ 3 & 5 \end{bmatrix}$

$A^t \cdot \vec{b} = \begin{bmatrix} -1 & 4 & 2 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -7 \\ 5 \\ -3 \\ 13 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 15 \end{bmatrix}$

d) $\det(A^t A) = 146$

$$(A^t A)^{-1} = \frac{1}{146}\begin{bmatrix} 5 & -3 \\ -3 & 31 \end{bmatrix}$$

$$(A^t A)^{-1} A^t b = \frac{1}{146}\begin{bmatrix} 5 & -3 \\ -3 & 31 \end{bmatrix}\begin{bmatrix} -5 \\ 15 \end{bmatrix}$$

$$= \frac{1}{146}\begin{bmatrix} -70 \\ 480 \end{bmatrix}$$

$$y = -\frac{35}{73}x + \frac{240}{73}$$

6) a) $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 20 \\ 15 \end{bmatrix} = 5 \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

yes, the eigenvalue is 5

using the definition

$$A \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 5 \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

b) only $\vec{u}$ and $\vec{v}$

since if $B \cdot \vec{v} = \lambda \vec{v}$,

$B \cdot \vec{v}$ is a multiple of $\vec{v}$,

and so on the same line

through the origin as $\vec{v}$.

- or -

a) $\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{bmatrix}\right)$

$= \lambda^2 - 3\lambda + 2 - 12$

$= \lambda^2 - 3\lambda - 10$

$= (\lambda - 5)(\lambda + 2)$

$\lambda = 5, -2$

since $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$

$= 5 \begin{bmatrix} 4 \\ 3 \end{bmatrix},$

$\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is an eigenvector.

7) a)

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

b)

$$B = \begin{bmatrix} 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/3 \\ 0 & 0 & 0 & 1/3 \\ 1 & 1/2 & 1/2 & 0 \end{bmatrix}$$

c)

$$C = \frac{17}{20} B + \frac{1 - 17/20}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/80 & 37/80 & 3/80 & 77/240 \\ 3/80 & 3/80 & 37/80 & 77/240 \\ 3/80 & 3/80 & 3/80 & 77/240 \\ 71/80 & 37/80 & 37/80 & 3/80 \end{bmatrix}$$

d)

$197813 + 175560 + 123200 + 325959$

$= 822532$

$$\frac{175560}{822532} \approx .2134$$

8) a)

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 6 \\ -10 \end{bmatrix}$$

$$\mathrm{Ran}(T) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -5 \\ 6 \\ -10 \end{bmatrix} \right\}$$

b) Note

$$\begin{bmatrix} -5 \\ 6 \\ -10 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix},$$

so $\mathrm{Ran}(T) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \right)$

Since $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is not a multiple

of $\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$, $\mathrm{Ran}(T)$ is a <u>plane</u>.

9) a) $\underbrace{\begin{bmatrix} 8 \\ 1 \\ 6 \end{bmatrix}}_{\vec{v}_2} = \frac{1}{2}\underbrace{\begin{bmatrix} 6 \\ 2 \\ 21 \end{bmatrix}}_{\vec{v}_1} + \frac{1}{2}\underbrace{\begin{bmatrix} 10 \\ 0 \\ -9 \end{bmatrix}}_{\vec{v}_3}$

so $\vec{v}_2 \in \text{span}\{\vec{v}_1, \vec{v}_3\}$

b) $\begin{bmatrix} 8 & 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 0 \\ -9 \end{bmatrix}$

$= 80 - 54 \neq 0$

so $\vec{v}_2$ and $\vec{v}_3$ are

not orthogonal.

use Gram-Schmidt!

$$\vec{w}_1 = \vec{v}_3$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2^{\,t}\vec{v}_3}{\vec{v}_3^{\,t}\vec{v}_3}\vec{v}_3$$

$$\vec{v}_3^{\,t}\vec{v}_3 = \begin{bmatrix} 10 & 0 & -9 \end{bmatrix}\begin{bmatrix} 10 \\ 0 \\ -9 \end{bmatrix}$$

$$= 100 + 81 = 181$$

$$\vec{w}_2 = \begin{bmatrix} 8 \\ 1 \\ 6 \end{bmatrix} - \frac{26}{181}\begin{bmatrix} 10 \\ 0 \\ -9 \end{bmatrix}$$

$$= \frac{1}{181}\begin{bmatrix} 1188 \\ 181 \\ 1320 \end{bmatrix}$$

use $181\,\vec{w}_2 = \begin{bmatrix} 1188 \\ 121 \\ 1320 \end{bmatrix}$

and $\vec{w}_1 = \begin{bmatrix} 10 \\ 0 \\ -9 \end{bmatrix}$

$$\|181\,w_2\|_2 = \sqrt{1188^2 + 121^2 + 1320^2}$$

$$= \sqrt{3168385}$$

$$= 11\sqrt{26185}$$

$$\|\vec{w}_1\|_2 = \sqrt{181}$$

$$\left\{ \underbrace{\frac{1}{\sqrt{181}} \begin{bmatrix} 10 \\ 0 \\ -9 \end{bmatrix}}_{\vec{v}_1},\ \underbrace{\frac{1}{11\sqrt{26185}} \begin{bmatrix} 1188 \\ 121 \\ 1320 \end{bmatrix}}_{\vec{v}_2} \right\}$$

is the ONB

c) $\vec{v}_1 \vec{v}_1^{\,t} = \frac{1}{181}\begin{bmatrix}10\\0\\-9\end{bmatrix}\begin{bmatrix}10 & 0 & -9\end{bmatrix}$

$$= \frac{1}{181}\begin{bmatrix}100 & 0 & -90\\0 & 0 & 0\\-90 & 0 & 81\end{bmatrix}$$

$$\vec{v}_2 \vec{v}_2^{\,t} = \frac{1}{3168385}\begin{bmatrix}1188\\121\\1320\end{bmatrix}\begin{bmatrix}1188 & 121 & 1320\end{bmatrix}$$

$$= \frac{1}{3168385}\begin{bmatrix}1411344 & 143748 & 1568160\\143748 & 14641 & 159720\\1568160 & 159720 & 1742400\end{bmatrix}$$

$P = \vec{v}_1 \vec{v}_1^{\,t} + \vec{v}_2 \vec{v}_2^{\,t}$ = a mess

10 a) No, not necessarily.

If $A^t \vec{v} = \vec{0}$, we know

A^t is not invertible,

so $\det(A^t) = 0$

But $\det(A^t) = \det(A)$,

so A is not invertible.

But A could be

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & & & & 0 \\ 0 & \cdots & & & & 0 \\ 0 & \cdots & & & & 0 \\ 0 & \cdots & & & & 0 \\ 0 & \cdots & & & & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

b) $\lambda = 0$

If $A^{-1}\vec{v} = 0 \cdot \vec{v}$ for some nonzero $\vec{v}$,
$= \vec{0}$

applying A,

$$\vec{v} = A \cdot (A^{-1}\vec{v}) = A \cdot \vec{0} = \vec{0}$$

This can't happen, so $\lambda = 0$
is not an eigenvalue.

11) a) $f(x) = 0,$

$f(x) = 2$

$f(x) = \pi$

b) $f(x) = \begin{cases} \frac{1}{x-42}, & x \neq 42 \\ 0, & x = 42 \end{cases}$

c) By a), W is nonempty.

Let $f, g \in W$. Then

$\lim_{x \to 42} g(x) = L$

$\lim_{x \to 42} f(x) = M$, and

$$\lim_{x \to 42} (f+g)(x) = \lim_{x \to 42} (f(x) + g(x))$$
$$= \lim_{x \to 42} f(x) + \lim_{x \to 42} g(x)$$
$$= M + L$$

So $f+g \in W$.

Now let $c \in \mathbb{R}$.

Then

$$\lim_{x \to 42} (c \cdot f)(x) = \lim_{x \to 42} c \cdot f(x)$$

$$= c \cdot \lim_{x \to 42} f(x)$$

$$= c \cdot M$$

So W is a subspace of $\mathcal{F}(\mathbb{R})$.

12) Let $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in S$.

$$2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$2^2 + 0^2 + 0^2 = 4 > 1,$$

So $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \notin S$.

Therefore, S is not a subspace of $\mathbb{R}^3$.