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W '23 Final

a) No, only 0, 1, or infinitely many solutions are possible

b) vector addition and scaler multiplication

c) points, lines, plane ($\mathbb{R}^2$)

d) point, line, plane, 3-D space ($\mathbb{R}^3$)

e) matrix

2) a) yes, 2×1

b) no

c) yes, 2×2

d) no

e) yes, 3×2

3 a) $y = ax^3 + bx^2 + cx + d$

$6 = d$

$2 = a + b + c + d$

$4 = 8a + 4b + 2c + d$

$3 = -a + b - c + d$

b) since $d = 6$, we get

$-4 = a + b + c$

$-3 = -a + b - c$

$-2 = 8a + 4b + 2c \Leftrightarrow -1 = 4a + 2b + c$

adding the first two equations,

$2b = -7$

$b = -7/2$

Subtracting the first two equations,

$$-1 = 2a + 2c$$

$$-\frac{1}{2} = a + c$$

$$c = -\frac{1}{2} - a$$

Substituting all this into the 3rd equation,

$$-1 = 4a + 2(-7/2) + (-1/2 - a)$$

$$\frac{13}{2} = 3a$$

$$a = 13/6$$

$$c = -1/2 - a = -8/3$$

c) $y = 13/6\, x^3 - \frac{7}{2} x^2 - 8/3\, x - 6$

- or -

b)

$$\begin{bmatrix} 1 & 1 & 1 & -4 \\ -1 & 1 & -1 & -3 \\ 2 & 4 & 8 & -2 \end{bmatrix}$$

$R1 + R2$

$$\begin{bmatrix} 1 & 1 & 1 & -4 \\ 0 & 2 & 0 & -7 \\ 2 & 4 & 8 & -2 \end{bmatrix}$$

$-2R1 + R3$

$$\begin{bmatrix} 1 & 1 & 1 & -4 \\ 0 & 2 & 0 & -7 \\ 0 & 2 & 6 & 6 \end{bmatrix}$$

$-R2 + R3$

$$\begin{bmatrix} 1 & 1 & 1 & -4 \\ 0 & 2 & 0 & -7 \\ 0 & 0 & 6 & 13 \end{bmatrix}$$

$-\frac{1}{2} R2 + R1$

$$\begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 2 & 0 & -7 \\ 0 & 0 & 6 & 13 \end{bmatrix}$$

$-\frac{1}{6} R3 + R1$

$$\begin{bmatrix} 1 & 0 & 0 & -8/3 \\ 0 & 2 & 0 & -7 \\ 0 & 0 & 6 & 13 \end{bmatrix}$$

4) a) $$\begin{bmatrix} \cos(-7\pi/6) & -\sin(-7\pi/6) & 0 \\ \sin(-7\pi/6) & \cos(-7\pi/6) & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{3}/2 & -1/2 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $$\begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) $$\begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

d) CAB

5) a) $y = mx + b$

$6 = b$

$2 = m + b$

$4 = 2m + b$

$3 = -m + b$

b) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 4 \\ 3 \end{bmatrix}$

c) $A^t A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$

$$A^t b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ 7 \end{bmatrix}$$

d) $$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} b \\ m \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 15 \\ 7 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 76 \\ -2 \end{bmatrix}$$

$$y = -\frac{1}{10} x + \frac{19}{5}$$

6) a) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

b) $\det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -37-\lambda & 14 \\ 0 & -105 & 40-\lambda \end{bmatrix}$

$$\begin{array}{ccccc} 1-\lambda & 0 & 0 & 1-\lambda & 0 \\ 0 & -37-\lambda & 14 & 0 & -37-\lambda \\ 0 & -105 & 40-\lambda & 0 & -105 \end{array}$$

$$= (1-\lambda)(-37-\lambda)(40-\lambda) + (105)(14)(1-\lambda)$$

$$0 = (1-\lambda)\left(\lambda^2 - 3\lambda + 105 \cdot 14 - 37 \cdot 40\right)$$

$$(\lambda^2 - 3\lambda - 10) = (\lambda-5)(\lambda+2)$$

$\lambda = 1$ or $\lambda = 5, -2$

c)

$$\begin{bmatrix} 0 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 10 \end{bmatrix}$$

7 a) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

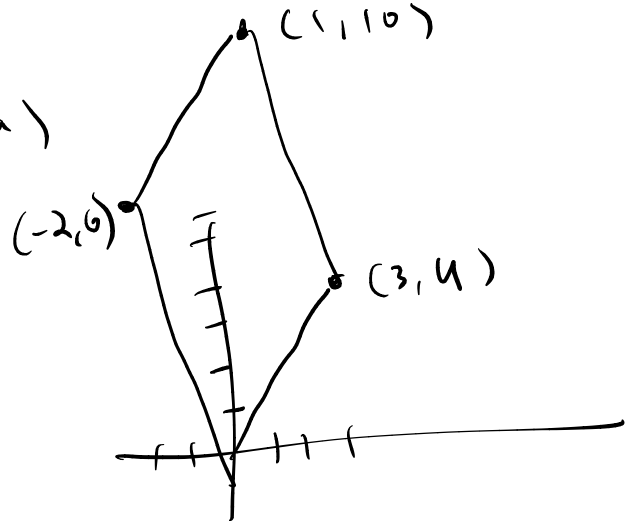
b) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1/2 & 1/3 \\ 0 & 0 & 1/3 \\ 1 & 1/2 & 1/3 \end{bmatrix}$

c) $17/20 \begin{bmatrix} 0 & 1/2 & 1/3 \\ 0 & 0 & 1/3 \\ 1 & 1/2 & 1/3 \end{bmatrix} + \dfrac{1-17/20}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/20 & 19/40 & 1/3 \\ 1/20 & 1/20 & 1/3 \\ 9/10 & 19/40 & 1/3 \end{bmatrix}$

d) $\dfrac{114}{80+80+114} = \dfrac{57}{40+40+57} = \dfrac{57}{137}$

8) a)



$$\text{Area} = \left| \det \begin{bmatrix} -2 & 3 \\ 6 & 4 \end{bmatrix} \right|$$

$$= |-8 - 18| = 26$$

b polar: angle:

$$\arctan\left(\frac{4}{-4}\right) + \pi$$

$$= \arctan(-1) + \pi$$

$$= -\pi/4 + \pi = 3\pi/4$$

$$x = r\cos(\theta) = \cos(3\pi/4 + \pi/6) = \cos\left(\frac{11\pi}{12}\right)$$

$$y = r\sin(\theta) = \sin(3\pi/4 + \pi/6) = \sin\left(\frac{11\pi}{12}\right)$$

$x \approx$

$y \approx$

9) a) $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\6\end{bmatrix}$ = 1st column

$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-4\\-8\end{bmatrix}$ = 2nd column

$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$ = 3rd column

$$A = \begin{bmatrix}3 & -4 & 1\\6 & -8 & 2\end{bmatrix}$$

b) $Ran(T) = Col(A)$

but all columns are multiples of $\begin{bmatrix}1\\2\end{bmatrix}$, so $\begin{bmatrix}x\\2x\end{bmatrix}$, $x \in \mathbb{R}$

c) Since all vectors are multiples of a nonzero vector, we get a <u>line</u>

d) $$\begin{bmatrix} 1 \\ 2 \end{bmatrix}^t \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5$$

$$P = \frac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}^t$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

10 a)

$$c v_1 + d v_3 = v_2$$

$$\begin{bmatrix} 0 \\ c \\ 6c \\ 7c \end{bmatrix} + \begin{bmatrix} -3d \\ 8d \\ 42d \\ 48d \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 10 \\ 8 \end{bmatrix}$$

1st entry: $-3d = -1$

$d = 1/3$

2nd entry: $c + 8/3 = 2$

$c = -2/3$

Last entry: $-14/3 + 16 \neq 8$

Not in span!

b)

$$w_1 = v_1 = \begin{bmatrix} 0 \\ 1 \\ 6 \\ 7 \end{bmatrix}$$

$$w_2 = v_2 - \frac{v_2^{t} v_1}{v_1^{t} v_1} v_1$$

$$= \begin{bmatrix} -1 \\ 2 \\ 10 \\ 8 \end{bmatrix} - \frac{118}{86} \begin{bmatrix} 0 \\ 1 \\ 6 \\ 7 \end{bmatrix}$$

$$\frac{59}{43}$$

$$= \begin{bmatrix} -1 \\ 27/43 \\ 76/43 \\ -69/43 \end{bmatrix}$$

$$= \frac{1}{43} \begin{bmatrix} -43 \\ 27 \\ 76 \\ -69 \end{bmatrix}$$

$$w_3 = v_3 - \frac{v_3^t v_1}{v_1^t v_1} \begin{bmatrix} 0 \\ 1 \\ 6 \\ 7 \end{bmatrix} - \frac{v_3^t w_2}{v_2^t w_2} \frac{1}{43} \begin{bmatrix} -43 \\ 27 \\ 76 \\ -64 \end{bmatrix}$$

$$w_3 = \begin{bmatrix} -3 \\ 8 \\ 42 \\ 48 \end{bmatrix} - \frac{546}{86} \begin{bmatrix} 0 \\ 1 \\ 6 \\ 7 \end{bmatrix} - \text{dogserel}$$

11 a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

b) $\det(ABA^{-1})$

$= \det(A) \cdot \det(B) \cdot \det(A^{-1})$

$= \det(A) \cdot 0 \cdot \det(A^{-1})$

$= 0$, so ABA^{-1} is not invertible

- or -

$\exists$ nonzero $\vec{v} \in \mathbb{R}^7$ with

$B\vec{v} = \vec{0}$

Let $\vec{w} = A \cdot v$

Then

$ABA^{-1} \cdot \vec{w}$

$= ABA^{-1} \cdot (A \cdot v)$

$= AB \cdot (A^{-1} \cdot A) v$

$= A \cdot B \cdot v$

$= A \cdot \vec{0} = \vec{0}$

So $w \in \ker(ABA^{-1})$, $w \neq \vec{0}$

so ABA^{-1} is not invertible

12 a) zero polynomial,

$x-1$

x^2-1

b) nonzero constant $p(x)=5$

c) By a), W is nonempty.

Let $p(x), q(x) \in W$. Assume

$\deg(p(x)) = n \geq \deg(q(x)) = m$

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad q(x) = \sum_{j=0}^{m} c_j x^j$$

$$q(x) + p(x) = \sum_{j=0}^{m} (a_j + c_j) x^j + \sum_{j=m+1}^{n} a_j x^j$$

adding up the coefficients,

$$\sum_{j=0}^{m} (a_j + c_j) + \sum_{j=m+1}^{n} a_j$$

$$= \sum_{j=0}^{n} a_j + \sum_{j=1}^{m} c_j$$

$$= 0 + 0 = 0 \checkmark$$

Now if $c \in \mathbb{R}$ is any constant,

$$c \cdot \sum_{j=0}^{n} a_j x^j = \sum_{j=0}^{n} (c \cdot a_j) x^j$$

Adding up the coefficients,

$$\sum_{j=0}^{n} c \cdot a_j$$

$$= c \sum_{j=0}^{n} a_j = c \cdot 0 = 0 \checkmark$$

So W is a subspace of

$\mathbb{P}[x]$