Determinants

(Section 3.2)

We know there are nonzero 2×2 matrices that are not invertible. We want a way to group these matrices together and quickly see whether they are invertible by calculating a single number: the determinant!

Example 1: (arbitrary 2x2 inverse)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Ask Wolfram Alpha for the inverse.

According to Wolfram Alpha, we get

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}$$

What if $ad-bc=0$?

Later...

The number $ad-bc$ is called the

determinant.

Picture:
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Multiply down diagonals, subtract the (upper right)·(lower left) from (upper left)·(lower right)

Q: What does is this number really saying, geometrically?

Example 2: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Compute $\det(A)$ and A^{-1}, if possible.

Solution: $\det(A) = 1 \cdot 3 - 2 \cdot 2 = -1$.

According to the formula,

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} [1\ \ 2]\cdot\begin{bmatrix} -3 \\ 2 \end{bmatrix} & [1\ \ 2]\cdot\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ [2\ \ 3]\cdot\begin{bmatrix} -3 \\ 2 \end{bmatrix} & [2\ \ 3]\cdot\begin{bmatrix} 2 \\ -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -3+4 & 2-2 \\ -6+6 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Example 3: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Compute $\det(A)$ and find A^{-1}, if possible.

Solution: $\det(A) = 1 \cdot 4 - 2 \cdot 2 = 0$

So the inverse formula doesn't apply.
How do we know A^{-1} won't exist?
Use HW3 written work:
find a nonzero vector $\vec{v}$ with $A\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then we will know A is not invertible.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+2 \\ -4+4 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So A is not invertible.

Note: $\begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} -b \\ a \end{bmatrix} = 0$

for all choices of a, b.

Geometric Interpretation

Back to dot products!

Recall: If $[v_1 \; v_2 \cdots v_n] = \vec{v}$ is a row vector and $\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \vec{w}$ is a column vector, then the dot product $\vec{v} \cdot \vec{w}$ is

$$\sum_{i=1}^{n} v_i \cdot w_i = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

Notation: $(\|\vec{w}\|_2)$. If $\vec{w}$ is either a column vector, we denote

$$\|\vec{w}\|_2 = \sqrt{w^t \cdot w} = \left(\sum_{i=1}^{n} w_i^2\right)^{1/2}$$

$$= \text{magnitude of } \vec{w}.$$

If $\vec{v}$ is a row vector, we get

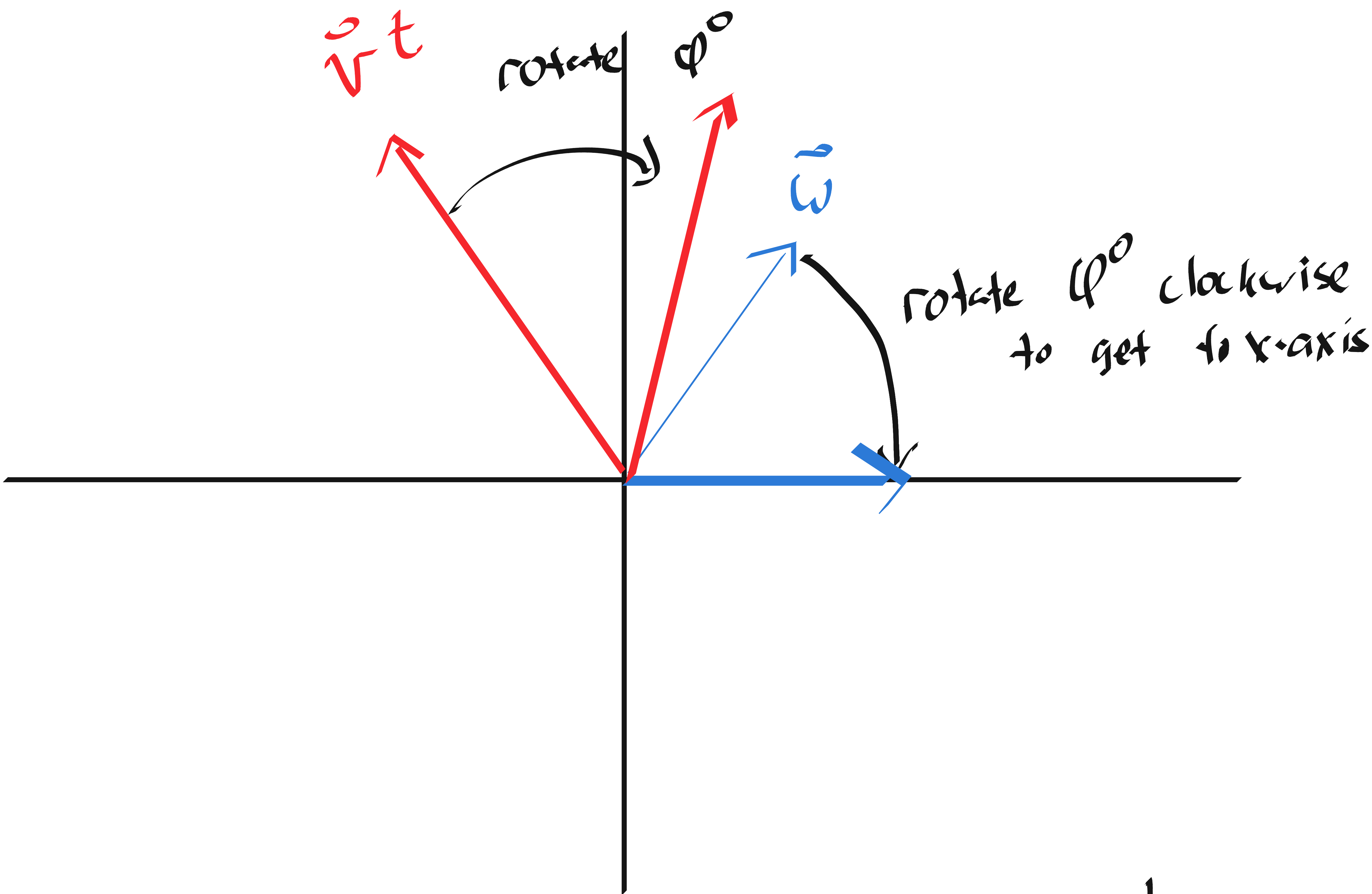
$$\|\vec{v}\|_2 = \sqrt{v \cdot v^t}$$

Reduction to $\mathbb{R}^2$

Observe that if $\vec{v}^{\,t}$ and $\vec{w}$ are vectors in $\mathbb{R}^n$ and they are **not** scalar multiples of each other, then their linear combinations (span) generate a plane. If they are multiples, their linear combinations generate a line if one vector is nonzero. Since we can do calculations in a plane, we will reduce questions about the dot product to $\mathbb{R}^2$.

Step 1: Use rotations to move one vector to the x-axis

Picture:



This preserves the angle between $\vec{v}^t$ and $\vec{w}$.

It also preserves the dot product!

Why this works, algebraically: We know counter-clockwise rotations are implemented by

$$A_\varphi = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

Clockwise would counter-clockwise by $-\varphi$

$$A_{-\varphi} = \begin{bmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{bmatrix}$$

$$A_{-\varphi} = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

Also note: $A_{-\varphi} = A_\varphi^{-1} = A_\varphi^t$

Compute

$$\left(A_{-\varphi}\vec{v}^{\,t}\right)^t \cdot \left(A_{-\varphi}\vec{\omega}\right)$$

$$= \left(\vec{v}^{\,t}\right)^t A_{-\varphi}^t A_{-\varphi}\vec{\omega}$$

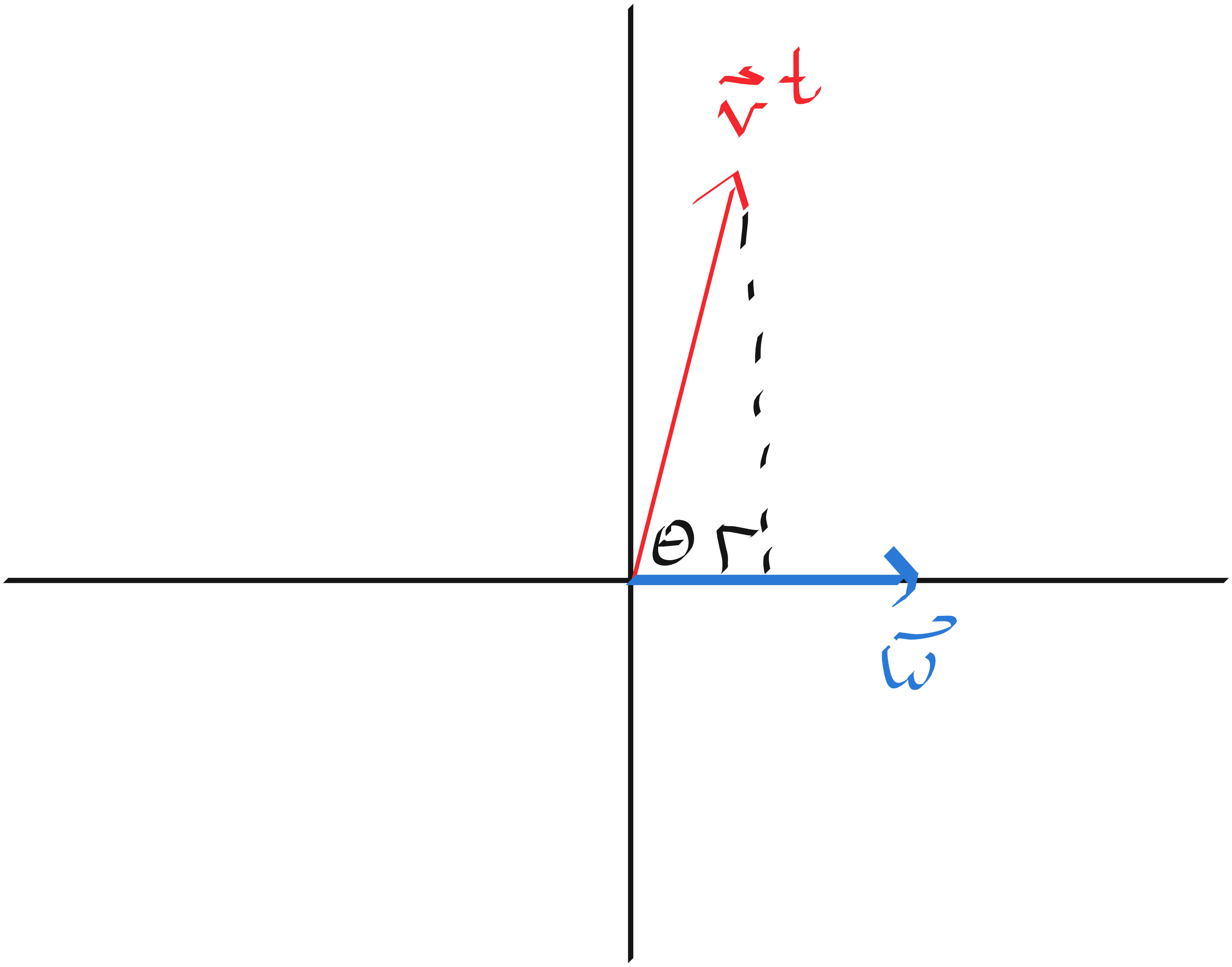
(transpose property $(AB)^t = B^t A^t$)

$$= \vec{v} \cdot \underbrace{A_{-\varphi}^t A_{-\varphi}}_{I_2} \vec{\omega}$$

since $A_\varphi^{-1} = A_\varphi^t$

$$= \vec{v} \cdot \vec{\omega}$$. This says rotations preserve dot products. So I can assume $\vec{\omega}$, say, is on the x-axis.

Step 2: Draw a picture



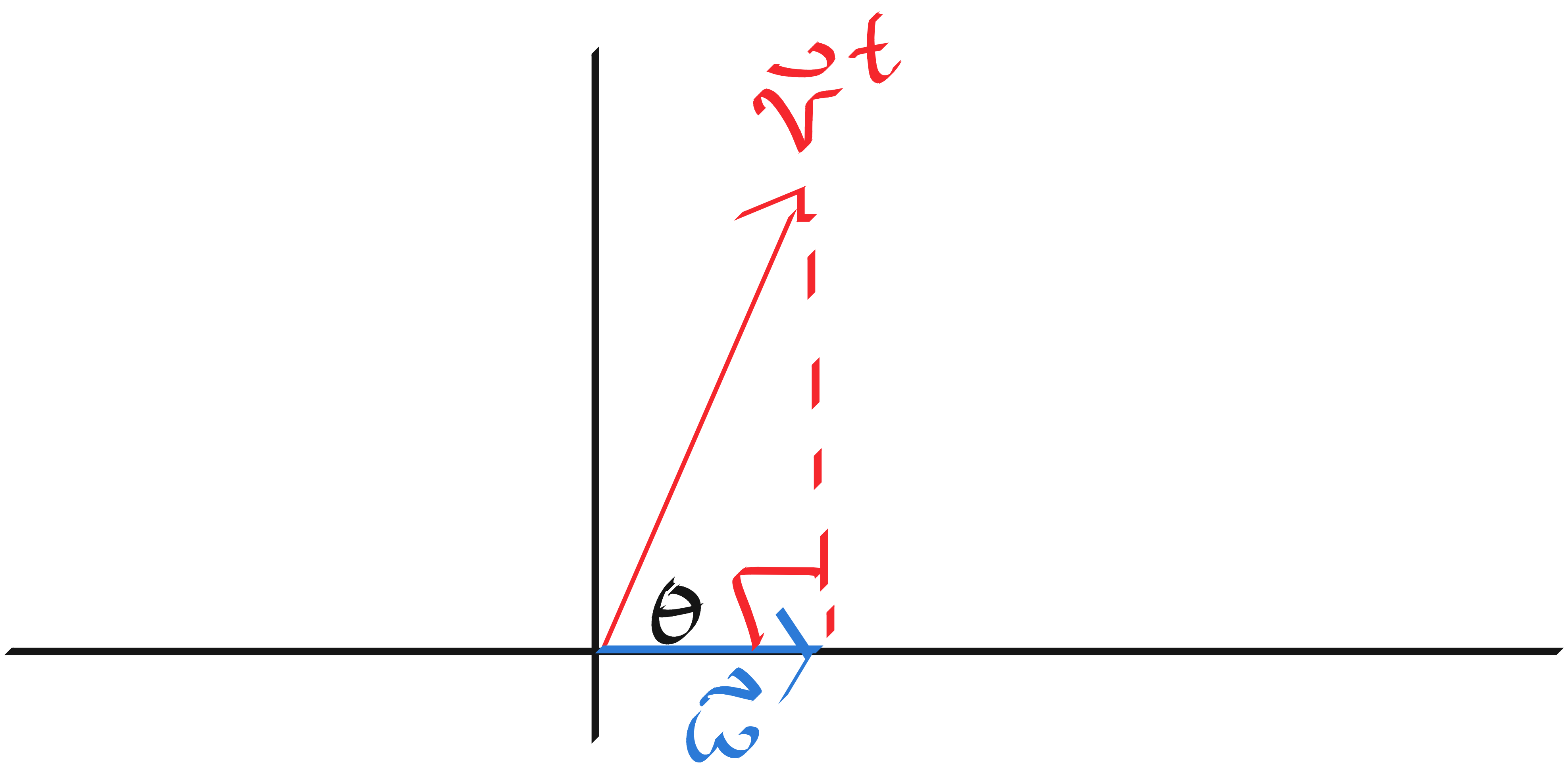
θ = angle between $\vec{v}^t$ and $\vec{w}$.

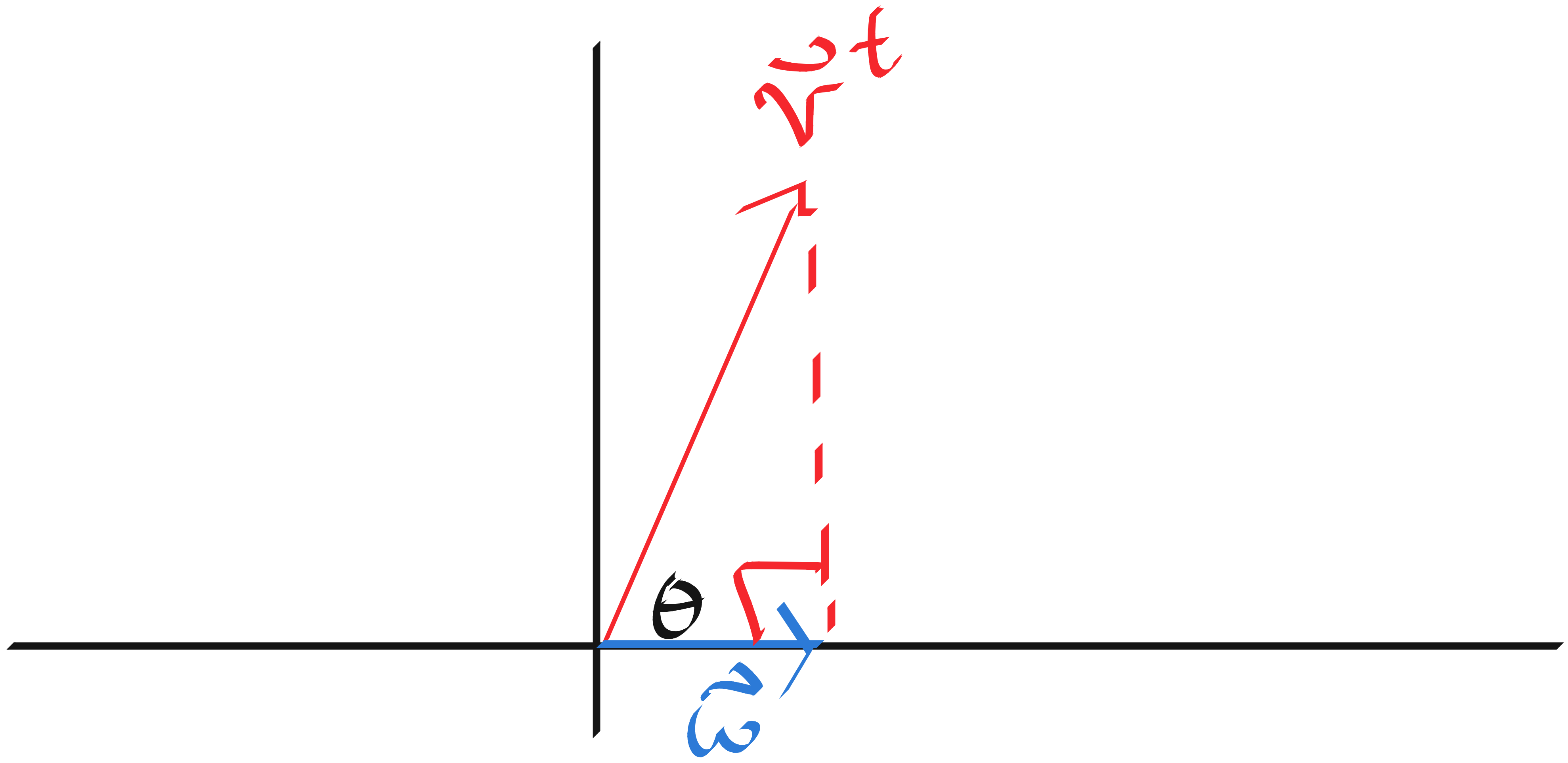
Find a formula for θ involving dot products!

We know that $\vec{w}$ is on the x-axis, so we can write

$$\vec{w} = \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ for a nonzero } x.$$

Observe that the angle between $\vec{v}^t$ and $\vec{w}$ is the same as the angle between $\vec{v}^t$ and any positive multiple of $\vec{w}$. We can then assume our picture looks like





$$\cos(\theta) = \frac{adj}{hyp}$$

If $\vec{\omega} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\vec{v}^t = \begin{bmatrix} x \\ y \end{bmatrix}$,

$$adj = x$$

$$hyp = \sqrt{x^2 + y^2}$$

So $\cos(\theta) = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{\|\vec{v}^t\|_2}$

Since $\vec{w} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and

$$\vec{v}^{\,t} = \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} x \\ 0 \end{bmatrix} = x^2$$

Similarly, $\|\vec{w}\|_2 = \sqrt{x^2+0^2} = x.$

Substituting,

$$\cos(\theta) = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\|\vec{v}^{\,t}\|_2} = \frac{x^2}{x \cdot \|\vec{v}^{\,t}\|_2}$$

$$\boxed{\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|_2 \|\vec{v}\|_2}}$$

Observe that if let $c > 0$,

the dot product formula would be

$$\frac{\vec{v} \cdot (c\vec{\omega})}{\|c\vec{\omega}\|_2 \|\vec{v}^t\|_2}$$ by substituting $c\vec{\omega}$ for $\vec{\omega}$.

$$= \frac{\cancel{c}\,(\vec{v} \cdot \vec{\omega})}{\cancel{c}\,\|\vec{\omega}\|_2 \|\vec{v}^t\|_2}$$

$$= \frac{\vec{v} \cdot \vec{\omega}}{\|\vec{\omega}\|_2 \|\vec{v}^t\|_2} = \cos(\theta)$$

This formula works for arbitrary vectors in $\mathbb{R}^n$:

if $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^n$, the angle θ between them is given by

$$\boxed{\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|_2 \|\vec{v}\|_2}}$$

General idea: dot products give you angles between vectors

Back to Determinants:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Suppose $\det(A) = ad - bc \neq 0$.

Then no column of A can be a multiple of the other column, since if

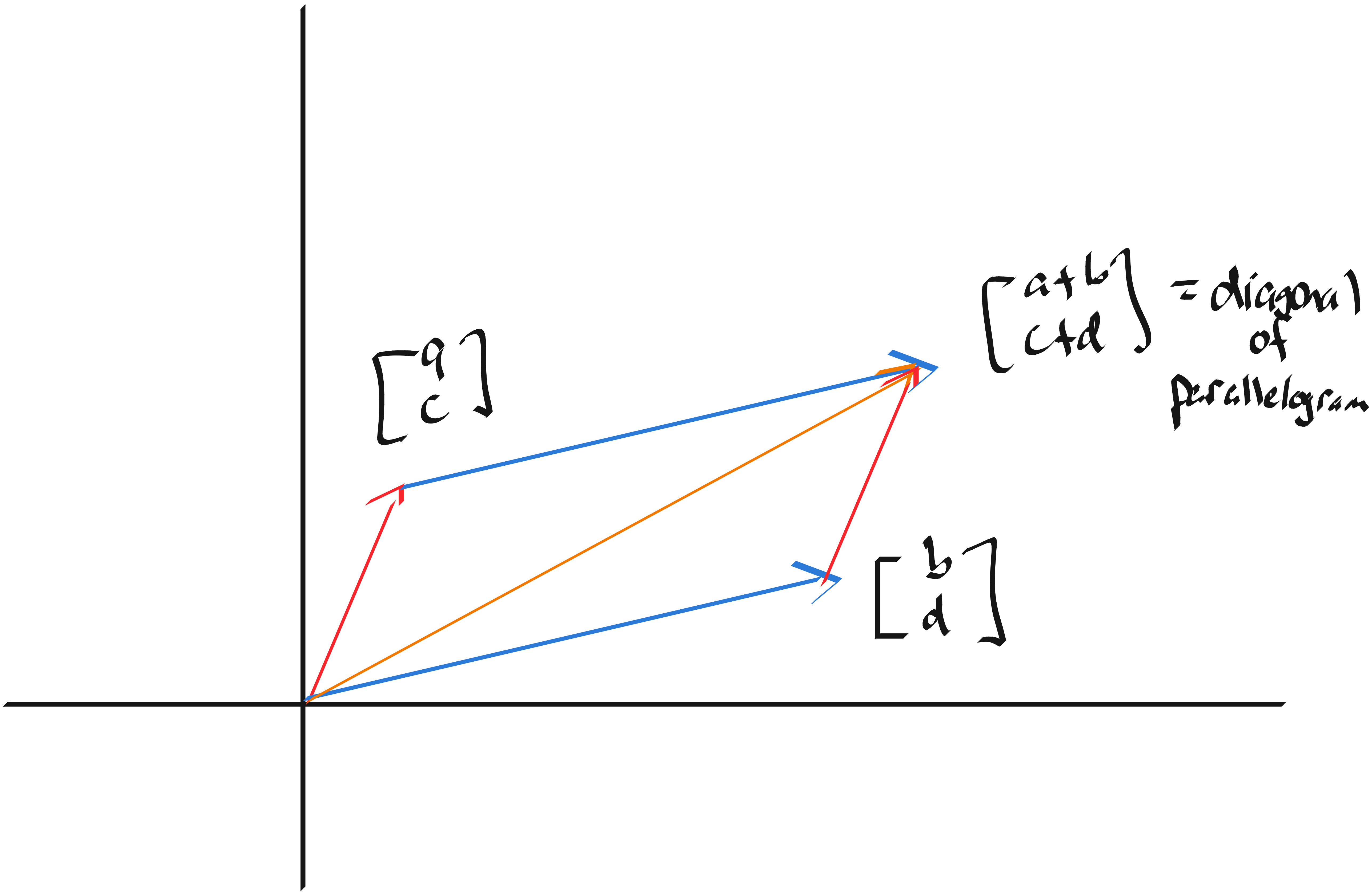
$$A = \begin{bmatrix} a & ka \\ c & kc \end{bmatrix} \quad \text{or}$$

$$A = \begin{bmatrix} kb & b \\ kd & d \end{bmatrix},$$

then $\det(A) = 0$.

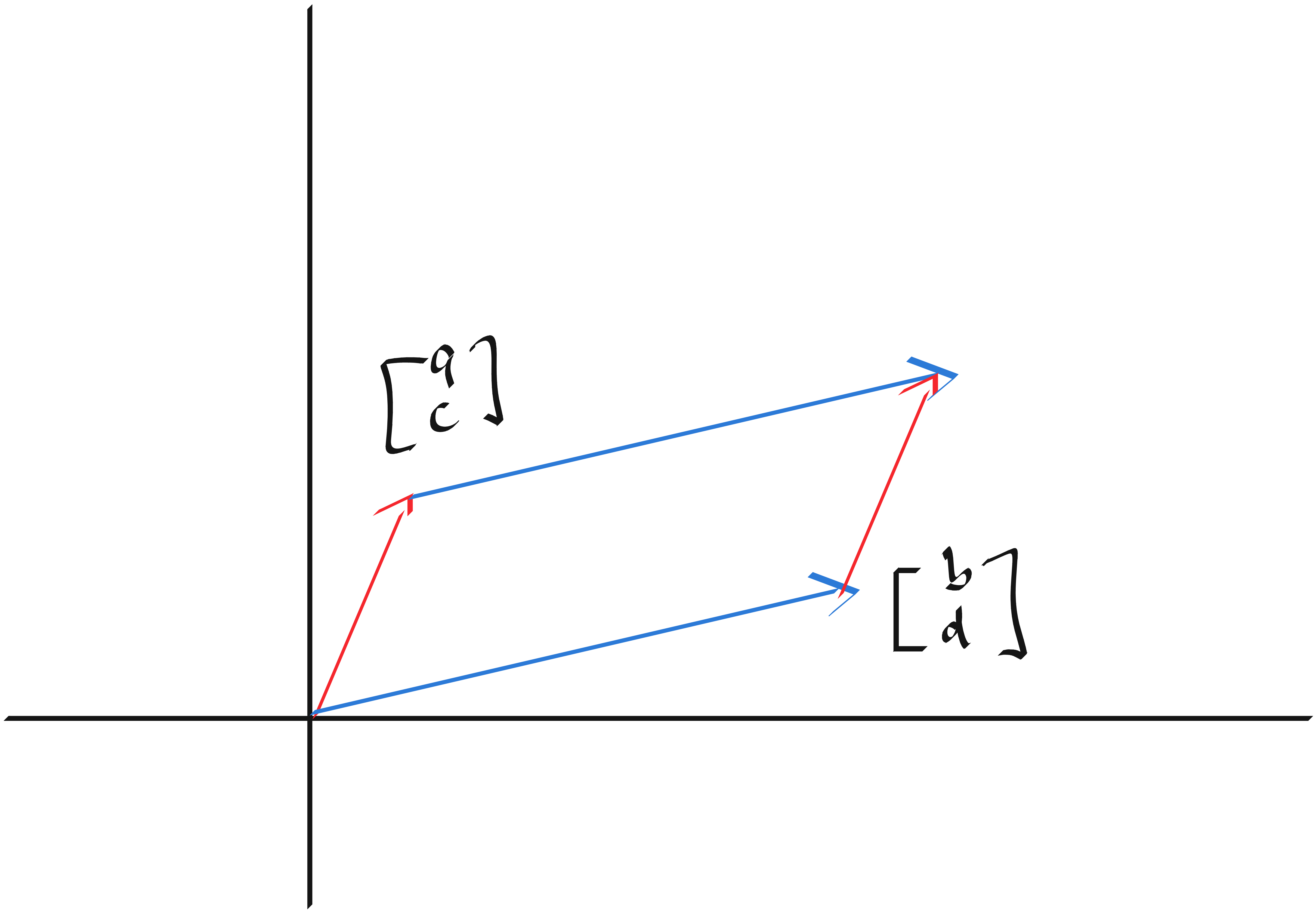
(k is a scalar)

Plot the Columns.



By adding $\begin{bmatrix} a \\ c \end{bmatrix}$ to $\begin{bmatrix} b \\ d \end{bmatrix}$,

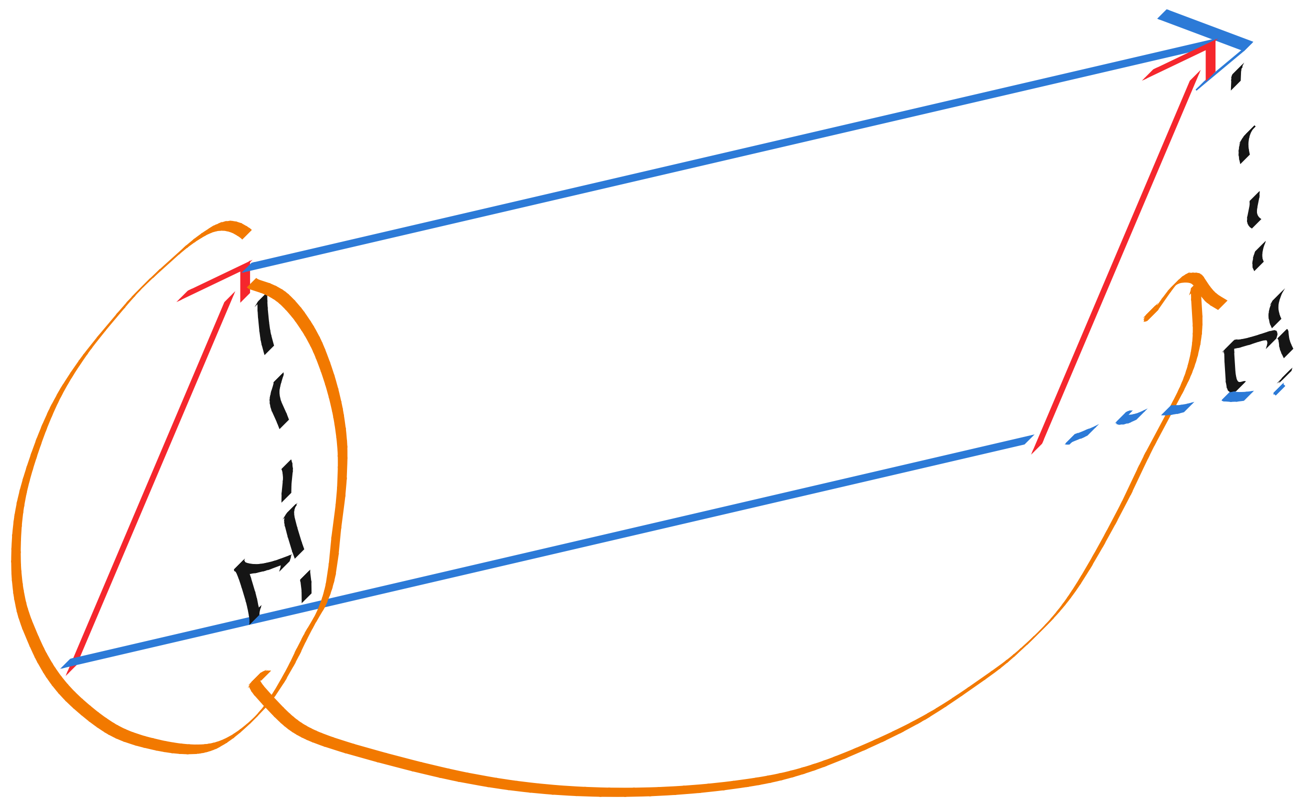
you get a parallelogram.



Compute the area of this parallelogram!

$$\text{Area} = (\text{width}) \cdot (\text{height})$$

because

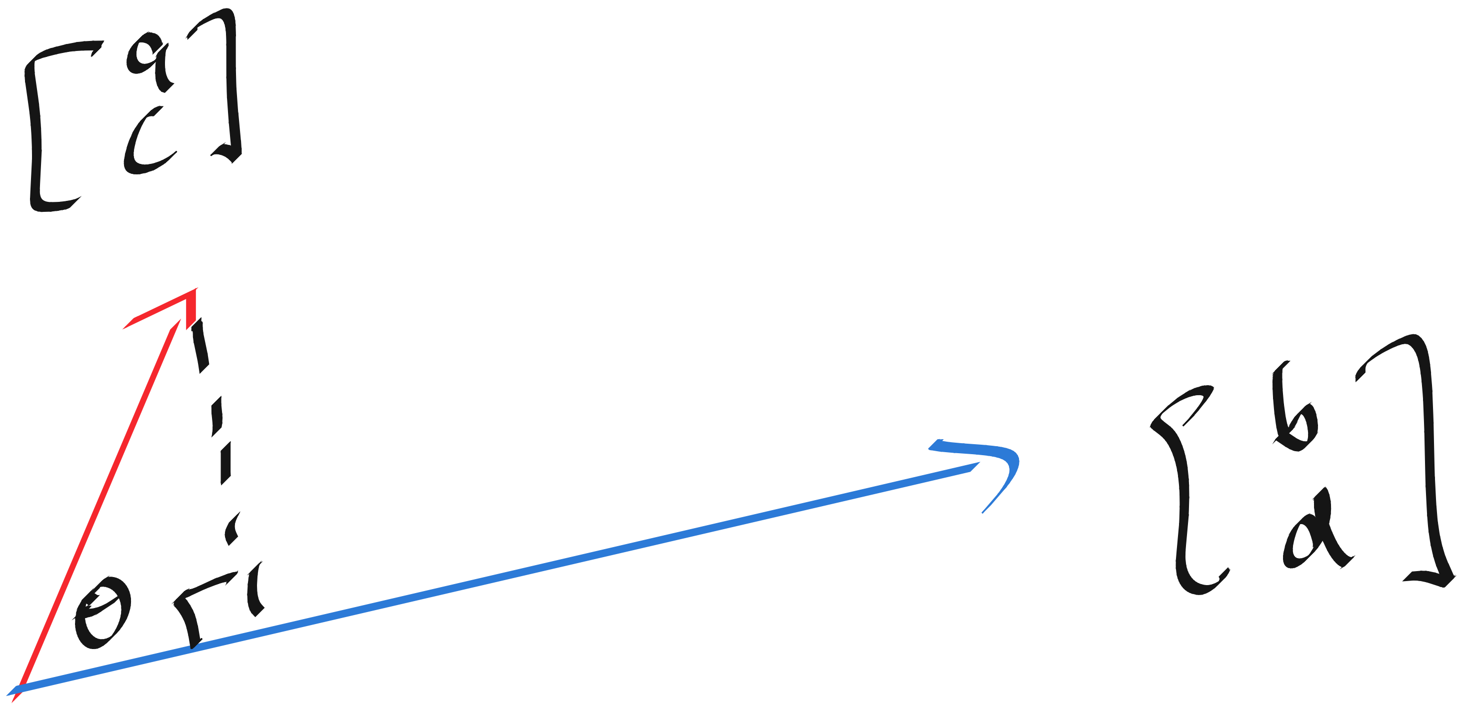


Cut out a right triangle, move it over to the other side, and we get a rectangle!

$$\text{Area} = (\text{width}) \cdot (\text{height})$$

$$\text{Width} = \left\| \begin{bmatrix} b \\ d \end{bmatrix} \right\|_2 = \sqrt{b^2 + d^2}$$

height?



Using the definition,

$$\cos(\theta) = \frac{adj}{hyp} = \frac{adj}{\left\| \begin{bmatrix} a \\ c \end{bmatrix} \right\|_2}$$

$$= \frac{adj}{\sqrt{a^2+c^2}}$$

$$adj = \cos(\theta)\sqrt{a^2+c^2}$$

use dot product formula

$$\text{adj} = \cos(\theta)\sqrt{a^2+c^2}$$

use dot product formula

$$\vec{v}^{\,t} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$\cos(\theta) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|_2 \, \|\vec{v}\|_2}$$

$$\cos(\theta) = \frac{\begin{bmatrix} a & c \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix}}{\|\vec{w}\|_2 \, \|\vec{v}\|_2}$$

$$\cos(\theta) = \frac{ab+cd}{\sqrt{b^2+d^2}\cdot\sqrt{a^2+c^2}}$$

Substitute

height = opp , but by the pythagorean theorem,

$$(opp)^2 + (adj)^2 = (hyp)^2$$

$$opp^2 + \frac{(ab+cd)^2}{b^2+d^2} = a^2+c^2$$

$$opp^2 = a^2+c^2 - \frac{(ab+cd)^2}{b^2+d^2}$$

make common denominator:

$$opp^2 = \frac{(a^2+c^2)(b^2+d^2) - (ab+cd)^2}{b^2+d^2}$$

$$\text{opp}^2 = \frac{(a^2+c^2)(b^2+d^2)-(ab+cd)^2}{b^2+d^2}$$

$$\text{opp}^2 = \frac{\cancel{a^2b^2}+c^2b^2+\cancel{c^2d^2}+a^2d^2-\cancel{a^2b^2}-\cancel{c^2d^2}-2abcd}{b^2+d^2}$$

$$\text{opp}^2 = \frac{\left.c^2b^2+a^2d^2-2abcd\right\}\text{ perfect square}}{b^2+d^2}$$

$$\text{opp}^2 = \frac{(ad-bc)^2}{b^2+d^2}.$$

Taking square roots,

$$\text{opp} = \frac{|ad-bc|}{\sqrt{b^2+d^2}}$$

(ad-bc could be negative)

Opp = height of parallelogram, so

Area = (width)(height)

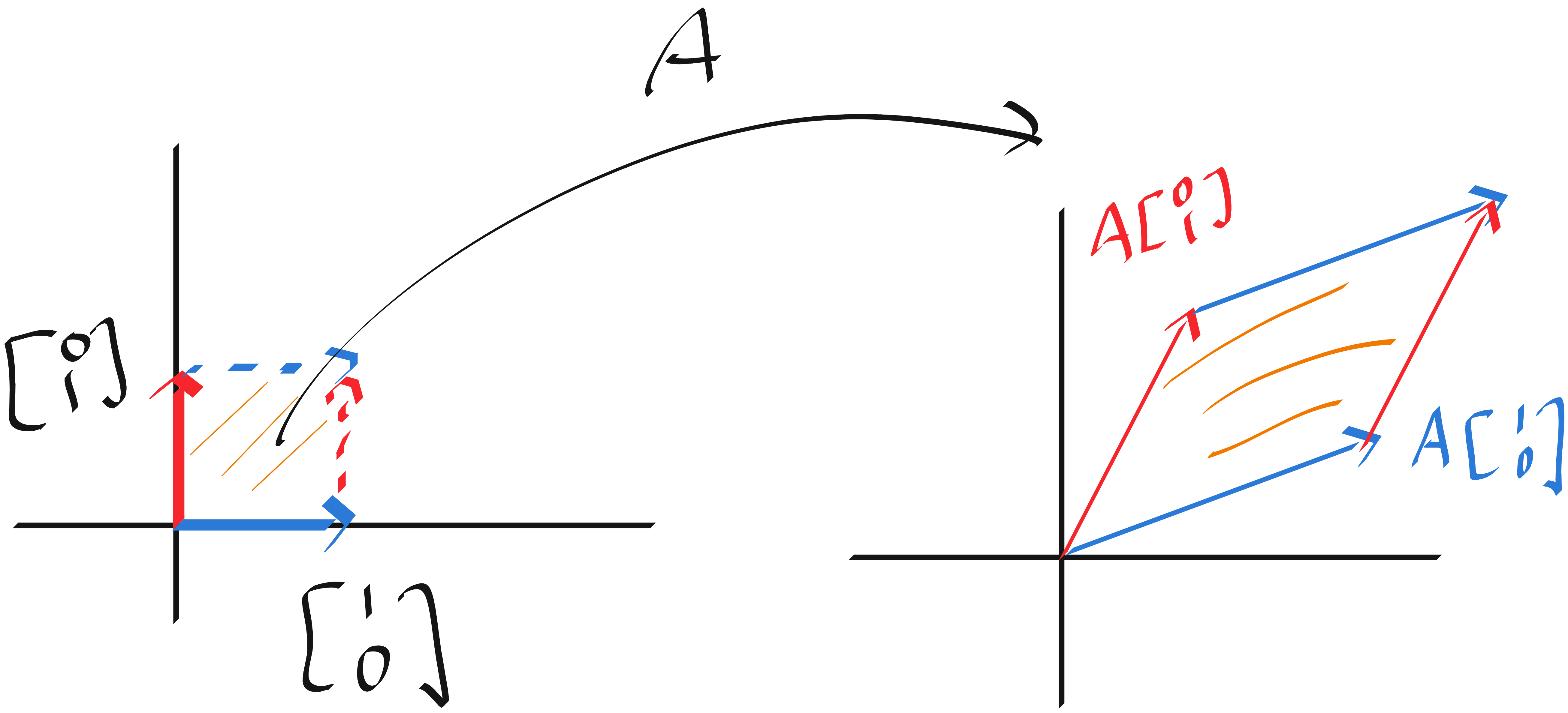
$$\text{Area} = \cancel{\sqrt{b^2+a^2}} \cdot \frac{|ad-bc|}{\cancel{\sqrt{b^2+a^2}}}$$

$$\boxed{\text{Area} = |ad-bc| = |\det(A)|}$$

So $|\det(A)|$ is the area of the parallelogram determined by the columns of A.

1st column of $A = A\begin{bmatrix}1\\0\end{bmatrix}$

2nd column of $A = A\begin{bmatrix}0\\1\end{bmatrix}$



Area of square = 1

Area of parallelogram $= |\det(A)|$

The matrix A is changing the area of the square, and $\det(A)$ tells you how much it is changing.

What if $\det(A) = 0$?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$ad - bc = 0$$

So $ad = bc$

If $a \neq 0$, then

$$d = \frac{bc}{a} = \left(\frac{b}{a}\right)c$$

and $b = \frac{a \cdot b}{a} = \left(\frac{b}{a}\right) \cdot a$

So the second column is a multiple of the first.

If $a=0$, then

$$A=\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$

and $ad-bc=0$, so since $a=0$,

$bc=0$.

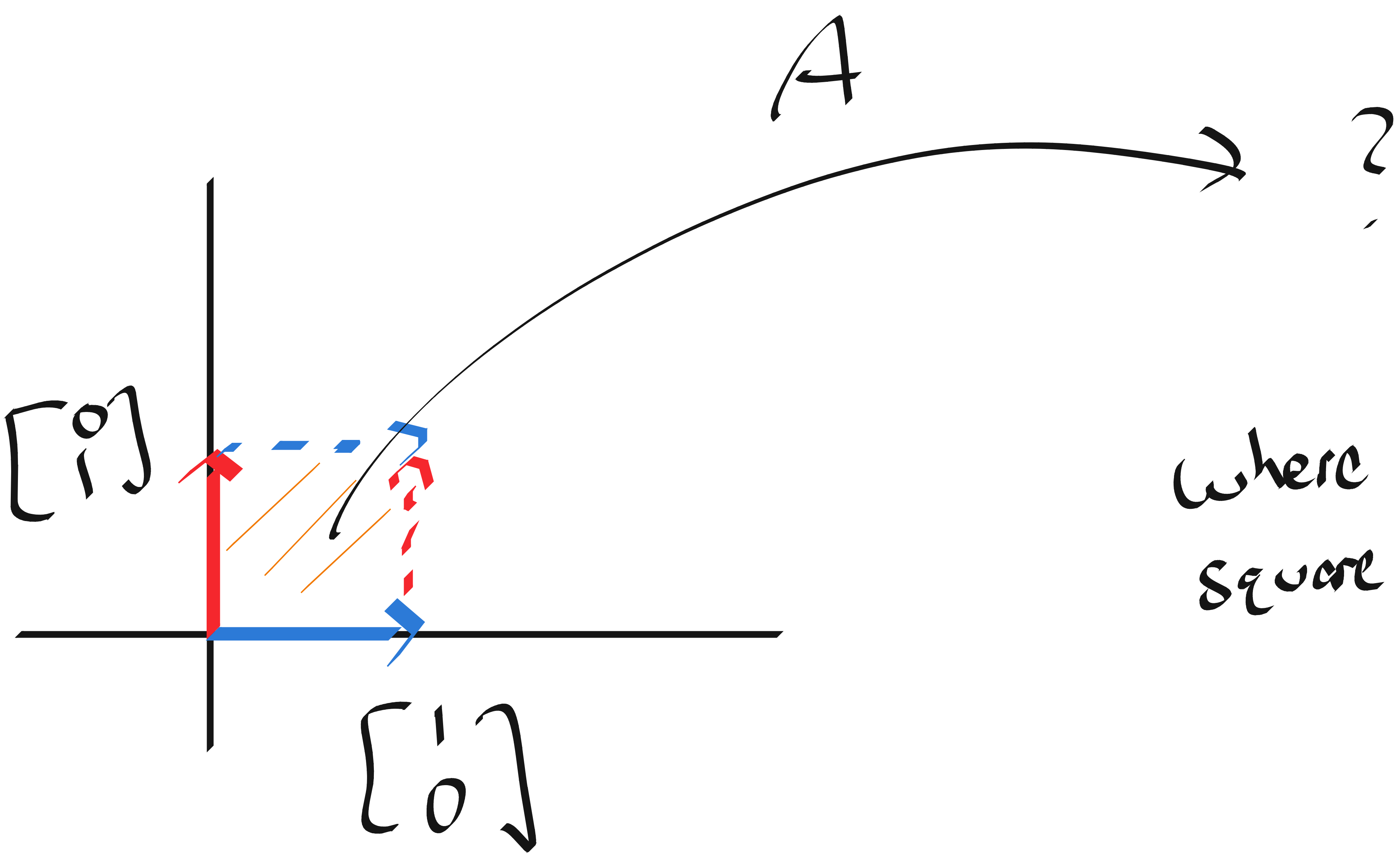
so either $\boxed{b=0 \text{ or } c=0}$.

If $c=0$,

$$A=\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$$

If $b=0$,

$$A=\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}.$$



where does the square go?

<u>Case 1:</u> $a \neq 0$. Then we know

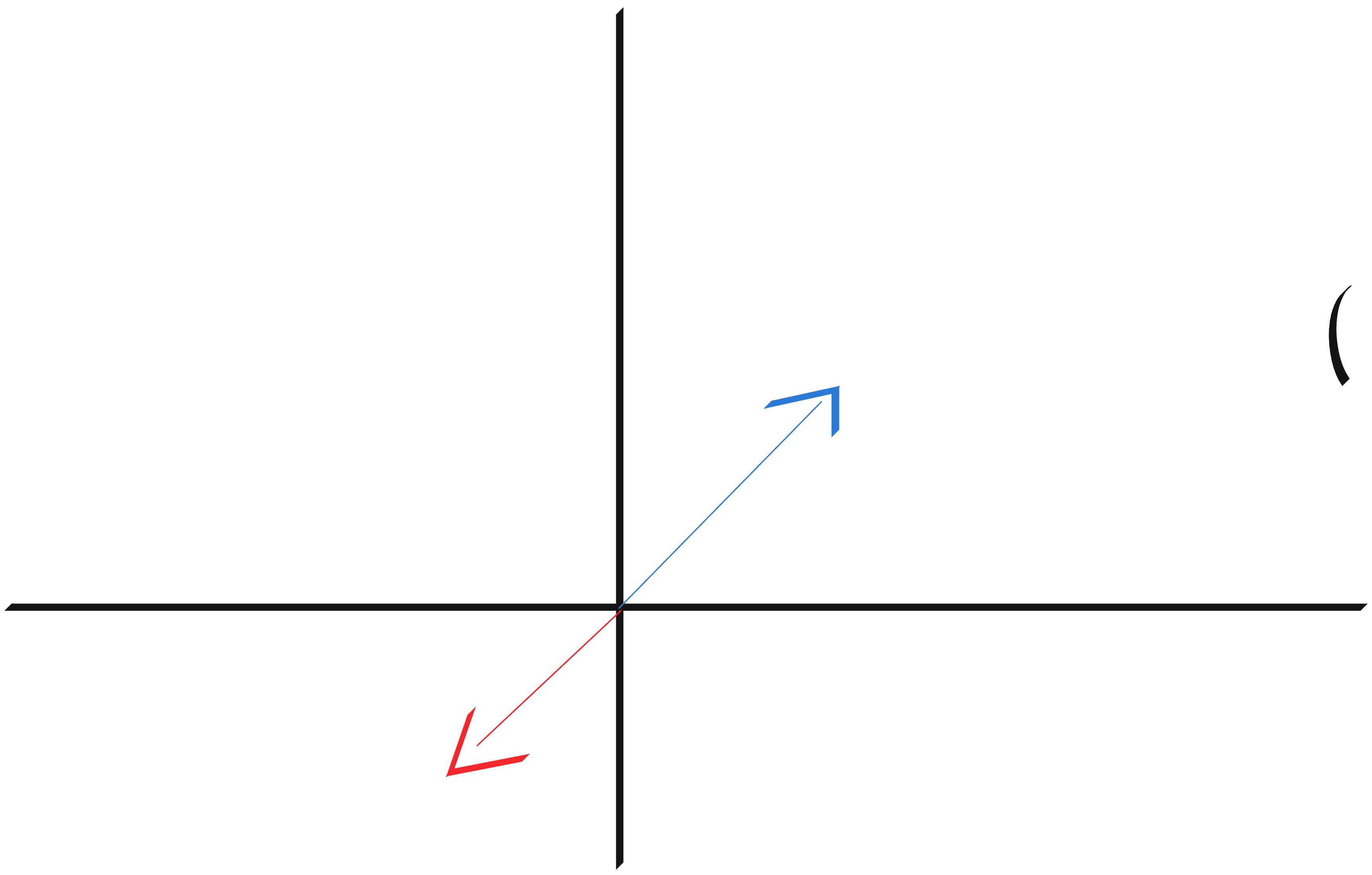
$$A = \begin{bmatrix} a & ka \\ c & kc \end{bmatrix}$$ for some

scalar k.

$$\begin{bmatrix} a \\ c \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

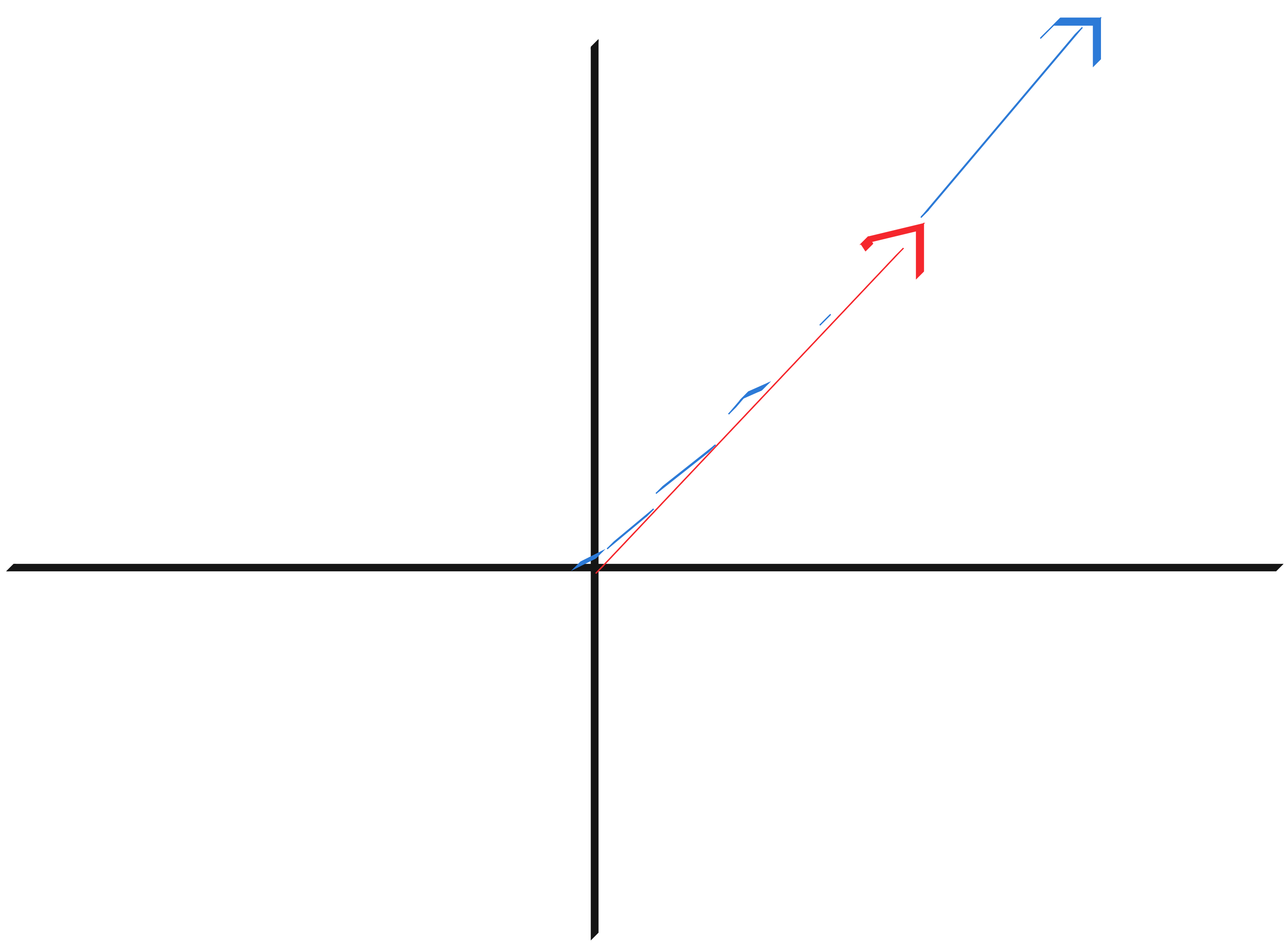
$$\begin{bmatrix} ka \\ kc \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$, scalar multiples!

So the square goes to either



$(k<0)$

or



$(k>0)$

No parallelogram, so no area!

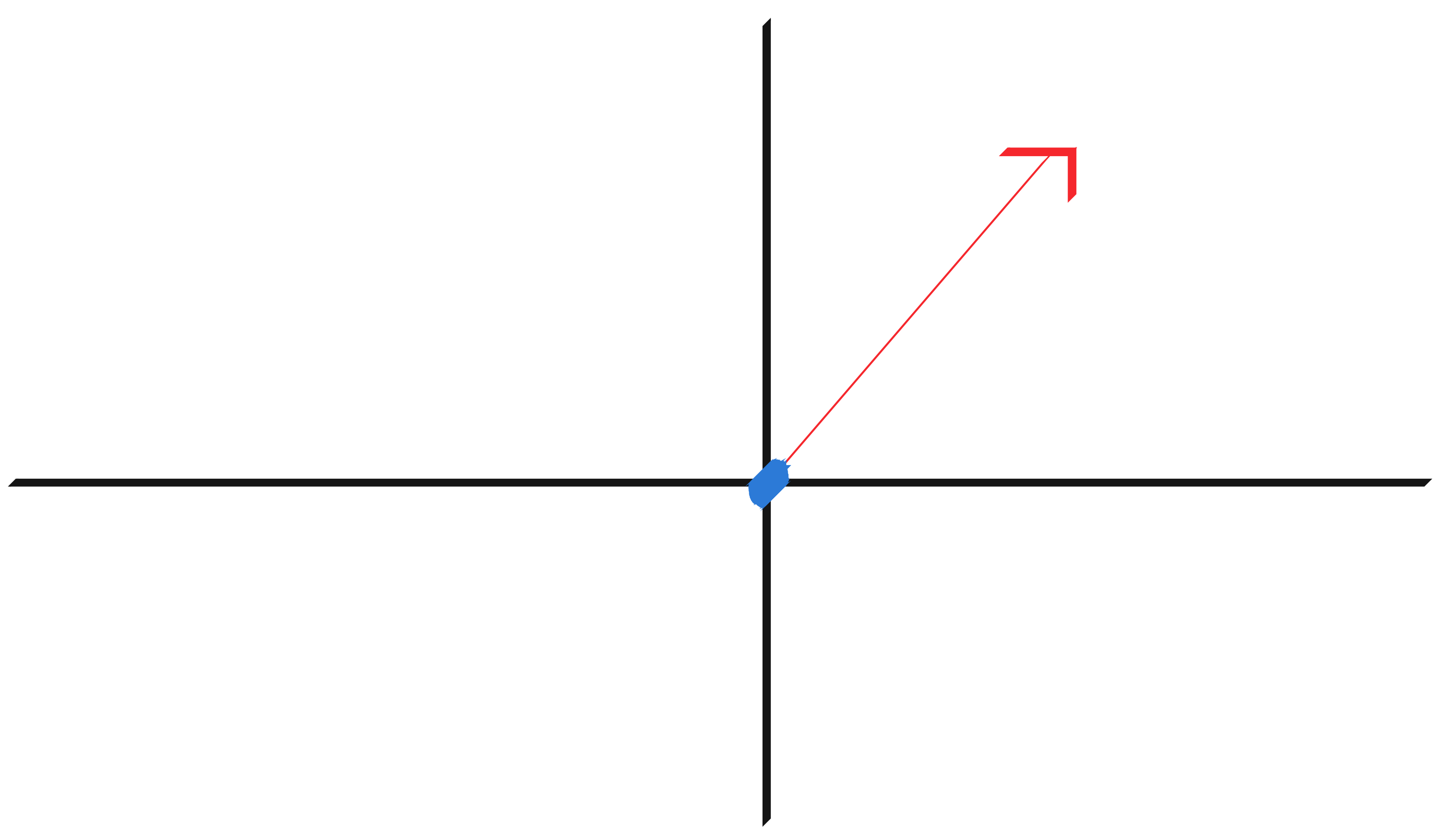
Case 2: $a=0$, $c=0$

$$A = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now our picture looks like



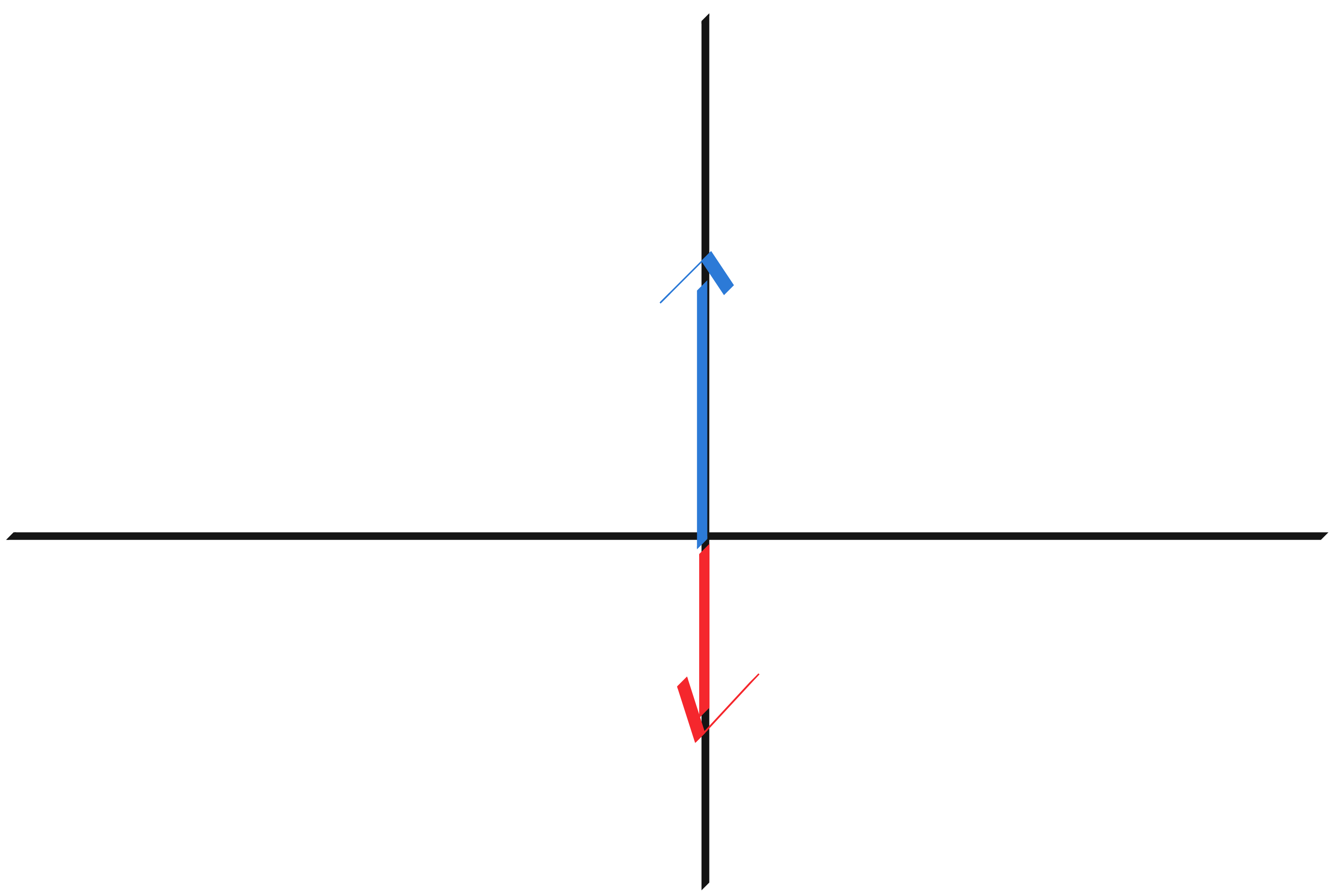
No area!

Case 3: $a=0,\ b=0$

$$A=\begin{bmatrix}0&0\\c&d\end{bmatrix}$$

$$\begin{bmatrix}0\\c\end{bmatrix}=A\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\d\end{bmatrix}=A\begin{bmatrix}0\\1\end{bmatrix}$$ again multiples of eachother!



Again, no area!

So if $\det(A) = 0$, the square gets crushed to a line segment or a point (zero matrix) and the resulting geometric shape has zero area.

Why det(A)=0 Means Noninvertibility

Lets take our 3 cases for

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc = 0$.

Case 1: $a \neq 0$

$$A = \begin{bmatrix} a & ka \\ c & kc \end{bmatrix}$$

$$\text{Let } \vec{v} = \begin{bmatrix} -k \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & ka \end{bmatrix} \begin{bmatrix} -k \\ 1 \end{bmatrix} = -ak + ak = 0$$

$$\begin{bmatrix} c & kc \end{bmatrix} \begin{bmatrix} -k \\ 1 \end{bmatrix} = -ck + ck = 0$$

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} -k \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & ka \\ c & kc \end{bmatrix} \begin{bmatrix} -k \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $\vec{v} = \begin{bmatrix} -k \\ 1 \end{bmatrix}$ is a nonzero vector with $A\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

By HW, A cannot be invertible.

Case 2: $a=0, c=0$

$$A = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1^{st} \text{ column of } A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So again, A cannot be invertible.

Case 3: $a=0, b=0$

$$A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$$

If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then

$A\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for any nonzero $\vec{v}$,

and so is not invertible.

If $A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then

$\vec{v} = \begin{bmatrix} -d \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives

$A\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So A is not invertible.

3x3 matrices

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Define $\boxed{\det(A) = aei + bfg + cdh - ceg - afh - bdi}$

Where did this come from?

Repeat the first two columns of A

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$

Multiply down diagonals, subtract the red products from the blue products.

Example 4:

Let $A = \begin{bmatrix} 1 & 5 & 6 \\ 3 & -2 & 4 \\ -11 & 8 & 0 \end{bmatrix}$

Compute $\det(A)$.

Solution: write

$$\begin{array}{ccccc} 1 & 5 & 6 & 1 & 5 \\ 3 & -2 & 4 & 3 & -2 \\ -11 & 8 & 0 & -11 & 8 \end{array}$$

$\det(A) = 0 - 220 + 144$

$\quad - 132 - 32 - 0$

$= -240$ ✓

General Definition

Permutations

A permutation on the numbers $1, 2, 3, \ldots, n$ is a function that assigns, to each whole number less that or equal to n, a unique whole number less than or equal to n.

Example 5: Pictures!

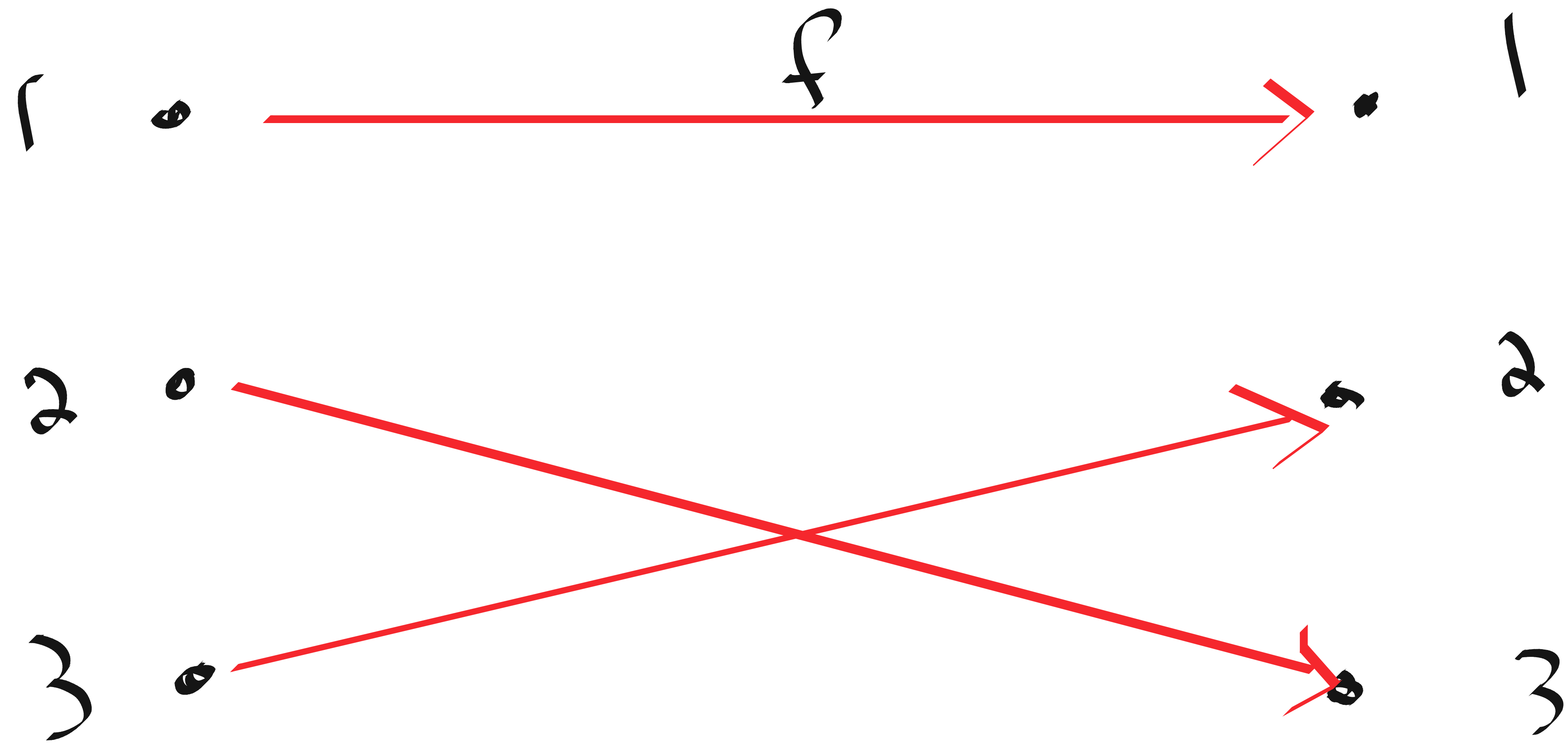
$n = 2$

$$\begin{tikzcd} 1 \arrow[r, "f"] & 1 \\ 2 \arrow[r] & 2 \end{tikzcd}$$

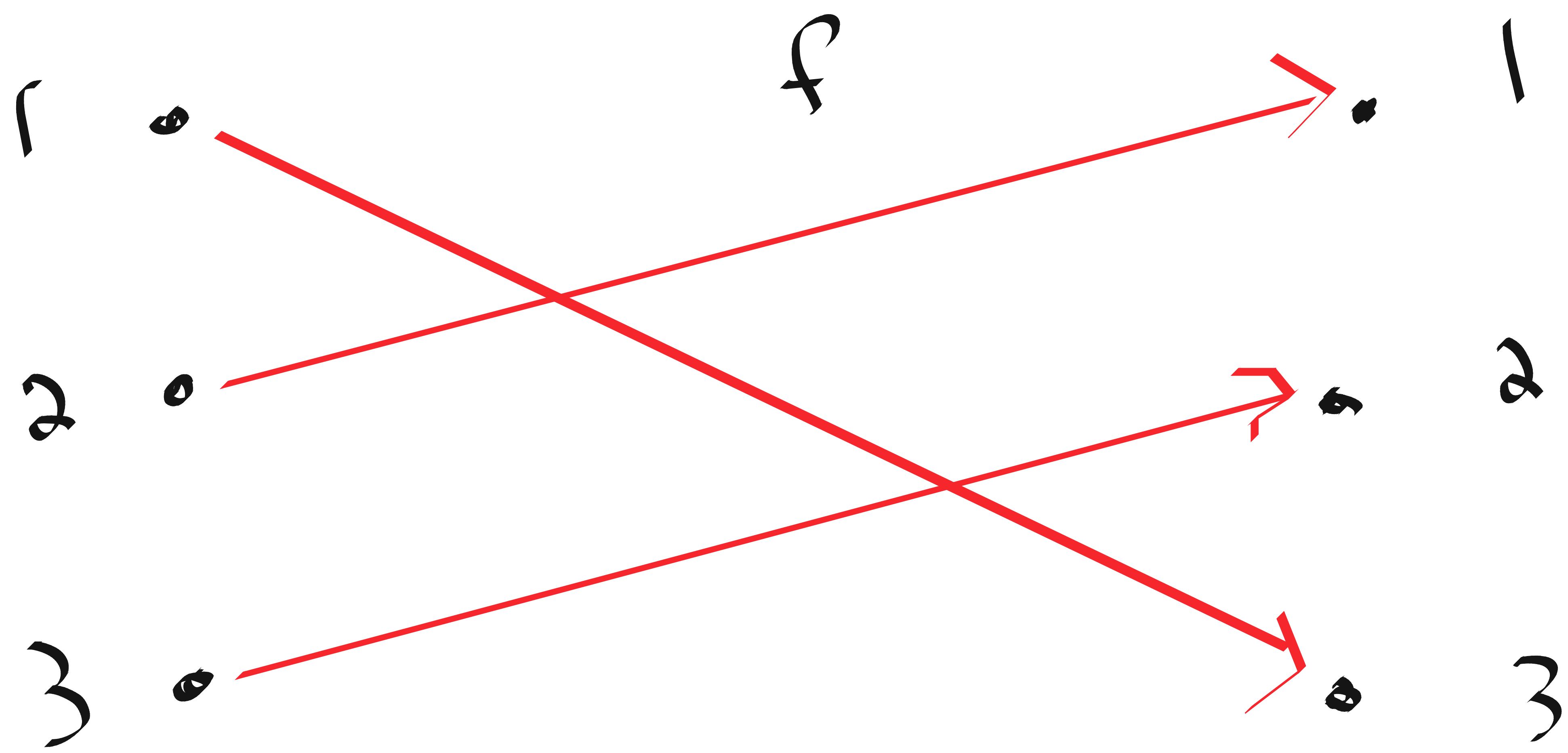
or

$$\begin{tikzcd} 1 \arrow[dr, "f"] & 1 \\ 2 \arrow[ur] & 2 \end{tikzcd}$$

$n = 3$



or



plus four other functions!

The Sign

Any permutation can be decomposed as a product of transpositions, where a transposition is a permutation that moves only two numbers. If f is a permutations, define

$$\text{Sign}(f) = (-1)^{\text{number of transpositions in } f}$$

So $\text{sign}(f)$ is either 1 or -1

General Definition

If A is an $n \times n$ matrix,

$A = (A_{i,k})_{i,k=1}^{n}$, then

if S_n denotes all permutations

of $\{1, 2, 3, \dots, n\}$, define

$$\boxed{\det(A) = \sum_{f \in S_n} \left(\operatorname{sign}(f) \cdot A_{1,f(1)} A_{2,f(2)} \cdots A_{n,f(n)} \right)}$$

Determinant Properties

Let A, B be $n \times n$ matrices

1) If A is diagonal, i.e.,

$A = (A_{i,k})_{i,k=1}^{n}$ and $A_{i,k} = 0$ if $i \neq k$,

then

$$\boxed{\det(A) = A_{1,1} \cdot A_{2,2} \cdot A_{3,3} \cdot \ldots \cdot A_{n,n}}$$

In particular,

$\det(I_n) = 1$

$\det(O_n) = 0$

2) $\det(AB) = \det(A) \cdot \det(B)$

3) A is invertible precisely when $\det(A) \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

4) $\det(A^t) = \det(A)$

You may use these properties freely and without penalty for the rest of this course!