Eigenvalues and Eigenvectors

(Section 3.3)

Wish: I wish every matrix was diagonal! Then finding inverses, determinants, products, etc. would be easy! Eigenvalues and eigenvectors help us get to this goal.

Eigenvalues and Eigenvectors

Let V be a vector space,

$T: V \to V$ a linear transformation.

Then a real number λ is an **eigenvalue** for T if there is a **nonzero** vector $x \in V$ with

$$\boxed{T(x) = \lambda x}$$

We call x an **eigenvector** associated to λ.

Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, we know there is an $n \times n$ matrix A with

$$T(\vec{x}) = A \cdot \vec{x} \text{ for}$$

all (column) vectors $\vec{x}$ in $\mathbb{R}^n$.

Because of this, the eigenvalues and eigenvectors of T will equal those of A, so we can always reduce to matrices in this case

Example 1: (diagonal matrix) Let

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Remember that if $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then

$$A\vec{e}_1 = 1^{st} \text{ column of } A = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} = 7\vec{e}_1$$

$$A\vec{e}_2 = 2^{nd} \text{ column of } A = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} = 8\vec{e}_2$$

$$A\vec{e}_3 = 3^{rd} \text{ column of } A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \vec{e}_3$$

We see that 7, 8, and 0 are eigenvalues for A, with associated eigenvectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$, respectively.

Observe: Zero can be an eigenvalue!

For diagonal matrices, we can always read off the eigenvalues from the diagonal entries, and the standard basis will always be eigenvectors.

Example 2: (non-diagonal matrix)

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

If we try the same trick as in Example 1,

$A\vec{e}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, not a multiple of $\vec{e}_1$

$A\vec{e}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, not a multiple of $\vec{e}_2$.

We are left trying to solve

$A\vec{x} = \lambda \vec{x}$ without knowing what we should choose for $\vec{x}$ or λ!

Determinants to the Rescue!

If $A\vec{x} = \lambda\vec{x}$, then

writing $\lambda\vec{x} = \lambda I_n \vec{x}$, then

$$A\vec{x} = \lambda I_n \vec{x}.$$

Subtracting $\lambda I_n \vec{x}$ from both sides,

$$A\vec{x} - \lambda I_n \vec{x} = \vec{0}, \text{ and}$$

factoring out an $\vec{x}$,

$$\boxed{(A - \lambda I_n)\vec{x} = \vec{0}}$$

If $(A-\lambda I_n)\vec{x}=\vec{0}$ for

some nonzero $\vec{x}$, then

$A-\lambda I_n$ is not invertible!

If $A-\lambda I_n$ is not invertible,

then $\boxed{\det(A-\lambda I_n)=0}$

To find eigenvalues for matrices,

we will just solve

$$\det(A-\lambda I_n)=0$$

for λ.

Back to Example 2

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A - \lambda I_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = (1-\lambda)(4-\lambda) - 2 \cdot 3$$

$$\det(A - \lambda I_2) = \lambda^2 - 5\lambda + 4 - 6$$

$$\det(A - \lambda I_2) = \lambda^2 - 5\lambda - 2.$$

Set equal to zero and solve for λ!

$$0 = \lambda^2 - 5\lambda - 2$$

Use the quadratic formula!

$$\lambda = \frac{5 \pm \sqrt{25 - 4(-2)}}{2}$$

$$\boxed{\lambda = \frac{5 \pm \sqrt{33}}{2}}$$

are the eigenvalues of A!

Check using Wolfram Alpha.

Just type in the matrix...

Example 3: (complex eigenvalues) Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Find the eigenvalues of A.

Solution: $$A - \lambda I_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I_2 = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}.$$

$$\det(A - \lambda I_2) = (-\lambda)^2 - (1)(-1)$$

$$\det(A - \lambda I_2) = \lambda^2 + 1$$

Setting equal to zero,

$$0 = \lambda^2 + 1$$

$$\lambda^2 = -1, \quad \text{so}$$

$$\boxed{\lambda = \pm i}$$

This shows that, even for matrices with real entries, complex numbers are unavoidable when it comes to eigenvalues.

Example 4: (3×3) $(1c)$, section 3.3)

$$A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$$

Find all eigenvalues of A.

Solution: $$A - \lambda I_3 = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I_3 = \begin{bmatrix} 7-\lambda & 0 & -4 \\ 0 & 5-\lambda & 0 \\ 5 & 0 & -2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_3) =$$

$$\begin{array}{ccccc} 7-\lambda & 0 & -4 & 7-\lambda & 0 \\ 0 & 5-\lambda & 0 & 0 & 5-\lambda \\ 5 & 0 & -2-\lambda & 5 & 0 \end{array}$$

$$= (7-\lambda)(5-\lambda)(-2-\lambda) - (-4)(5)(5-\lambda)$$

$$= (5-\lambda)\big((7-\lambda)(-2-\lambda) + 20\big)$$

$$= (5-\lambda)(\lambda^2 - 5\lambda - 14 + 20)$$

$$= (5-\lambda)(\lambda^2 - 5\lambda + 6)$$

$$= (5-\lambda)(\lambda-3)(\lambda-2)$$

Setting equal to zero,

$$0 = (5-\lambda)(\lambda-3)(\lambda-2),$$

so $\boxed{\lambda = 5, 3, 2}$

What about eigenvectors?

Supposing you know the value of an eigenvalue λ, then we can substitute it into

$$(A - \lambda I_n)\vec{x} = \vec{0}$$

and solve for $\vec{x}$. In the previous example, substituting $\lambda = 2$ gives

$$A - 2I_3 = \begin{bmatrix} 7-2 & 0 & -4 \\ 0 & 5-2 & 0 \\ 5 & 0 & -2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & -4 \\ 0 & 3 & 0 \\ 5 & 0 & -4 \end{bmatrix}$$

We solve

$$\begin{bmatrix} 5 & 0 & -4 \\ 0 & 3 & 0 \\ 5 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A - \lambda I_n$ is never invertible

if λ is an eigenvalue, so

you're left with row reduction:

$$\mathrm{RREF}\left(\left[\, A - \lambda I_n \;\middle|\; \vec{0} \,\right]\right).$$

For our example,

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 5 & 0 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 5 & 0 & -4 & 0 \end{array}\right]\right)$$

But Wolfram Alpha gives us eigenvectors without even asking, so why do this?

Example 5: Let $A = \begin{bmatrix} 3 & -7 \\ -7 & 3 \end{bmatrix}$.

The eigenvalues of A are $\lambda_1 = 10$ and $\lambda_2 = -4$.

Check that $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector associated to λ_1 and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector associated to λ_2.

Solution:

$$A - 10 I_2 = \begin{bmatrix} 3 & -7 \\ -7 & 3 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -7 \\ -7 & -7 \end{bmatrix}$$

$$(A - 10I_2)\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-7 & -7\\-7 & -7\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\end{bmatrix} \checkmark$$

Let's try a different method for $\lambda_2 = -4$:

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3 & -7\\-7 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \begin{bmatrix}-4\\-4\end{bmatrix}$$

$$= -4\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \lambda_2\begin{bmatrix}1\\1\end{bmatrix} \checkmark$$