Invertibility

(Section 2.4)

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix B such that

$$A \cdot B = B \cdot A = I_n$$

We will call an $n \times n$ matrix a square matrix.

Q: Why is this only for square matrices?

Suppose A is $m\times n$. Then

$$\underbrace{A}_{m\times n} \cdot \underbrace{B}_{n\times m} = I_m .$$

But if $m \neq n$,

$$\underbrace{B}_{n\times m} \cdot \underbrace{A}_{m\times n} = I_n$$

So $B\cdot A \neq A \cdot B$.

Bigger problem: if $m > n$, then B will multiply nonzero vectors and give the zero vector. Similarly, if $n > m$, the same phenomenon happens with A.

Since for any vector $\vec{v}$ in $\mathbb{R}^n$,

$I_n \cdot \vec{v} = \vec{v}$, this tells

you that A can never

satisfy $A \cdot B = I_m$ and

$B \cdot A = I_n$.

Further question: Is every nonzero $n \times 1$ matrix invertible?

Example 1:

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e_{1,2}$.

Is A invertible?

Solution: Suppose $B = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$ and

$$A \cdot B = B \cdot A = I_2.$$

$$\underline{A \cdot B = I_2}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} e & f \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $0 \neq 1$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is

<u>not</u> invertible!

So not even every nonzero $n \times n$ matrix is invertible. We'd like to be able to determine when an $n \times n$ matrix is invertible or not. For now, we are stuck with actually trying to exhibit an inverse.

Row Reduction and Inverses

Suppose A is an $n \times n$ invertible matrix:

Write $A^{-1} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$,

where each $\vec{v}_i$ is an $n \times 1$ column vector, $1 \le i \le n$.

Then since $\boxed{A \cdot A^{-1} = I_n}$

$$A \cdot \vec{v}_1 = 1^{st} \text{ column of } I_n = \vec{e}_1$$

$$A \cdot \vec{v}_2 = 2^{nd} \text{ column of } I_n = \vec{e}_2$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$A \cdot \vec{v}_n = n^{th} \text{ column of } I_n = \vec{e}_n$$

Writing as augmented matrices,

$$\left[\begin{array}{c|c} A & \vec{e}_1 \end{array}\right]$$

$$\left[\begin{array}{c|c} A & \vec{e}_2 \end{array}\right]$$

$$\vdots$$

$$\left[\begin{array}{c|c} A & \vec{e}_n \end{array}\right]$$

Row reduce!

Since the coefficient matrix is always A, we always do the same row reduction. So we can write all these augmented matrices simultaneously as

$$\left[A \;\vdots\; \vec{e}_1 \; \vec{e}_2 \cdots \vec{e}_n \right]$$

$$= \left[A \;\vdots\; I_n \right]$$

and row reduce this! If you get a solution, A is invertible, and the row-reduction will yield

$$\left[\begin{array}{c:c} I_n & A^{-1} \end{array}\right].$$

If A is not invertible, no solution exists.

Example 2: Find the inverse of

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 6 & 8 \\ 4 & 1 & 5 \end{bmatrix}, \text{ if}$$

it exists.

Solution: Start with a giant augmented matrix

$$[A \,\vdots\, I_3]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 2 & 6 & 8 & 0 & 1 & 0 \\ 4 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

Row reduce!

From Wolfram Alpha:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right. \left\vdots\ \underbrace{\begin{array}{ccc} -1/4 & -17/88 & 9/22 \\ -1/4 & 3/88 & 1/22 \\ 1/4 & 13/88 & -3/22 \end{array}}_{A^{-1}}\right]$$

A is invertible, with

$$\boxed{A^{-1} = \begin{bmatrix} -1/4 & -17/88 & 9/22 \\ -1/4 & 3/88 & 1/22 \\ 1/4 & 13/88 & -3/22 \end{bmatrix}}$$

Wolfram Alpha

Just type in the matrix, it will tell you whether the matrix is invertible or not, and find the inverse (if it can).

Invertibility and Systems of Linear Equations

Remember that, given a system of linear equations, we turned the system into a matrix-vector equation

$$A\vec{v} = \vec{b}$$

where A is the coefficient matrix, $\vec{v}$ is the column vector of variables, and $\vec{b}$ is the column vector of solutions.

We solved the system by row-reducing the augmented matrix

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right].$$

But if A is invertible, there is another way to solve: multiply both sides of $A\vec{v} = \vec{b}$ by A^{-1} on the left.

$$A^{-1} \cdot (A\vec{v}) = A^{-1} \cdot \vec{b}$$

$$(A^{-1} \cdot A) \cdot \vec{v} = A^{-1} \cdot \vec{b}$$

$$I_n \cdot \vec{v} = A^{-1} \cdot \vec{b}$$

Then $\vec{v} = A^{-1} \cdot \vec{b}$, and you can read off the solutions to the original linear system of equations using the components of $A^{-1} \cdot \vec{b}$.

Example 3: Find the interpolating cubic through the points $(1,-2)$, $(2,4)$, $(-1,6)$, and $(-3,19)$.

Solution: System of linear equations first.

Our interpolating cubic looks like

$$ax^3 + bx^2 + cx + d = y.$$

Plug in the points

$(1,-2)$: $a+b+c+d = -2$

$(2,4)$: $8a+4b+2c+d = 4$

$(-1,6)$: $-a+b-c+d = 6$

$(-3,19)$: $-27a+9b-3c+d = 19$

$$a+b+c+d=-2$$

$$8a+4b+2c+d=4$$

$$-a+b-c+d=6$$

$$-27a+9b-3c+d=19$$

Write as $A\vec{v}=\vec{b}$ where

$$\vec{v}=\begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}, \quad \vec{b}=\begin{bmatrix} -2 \\ 4 \\ 6 \\ 19 \end{bmatrix}$$

$$A=\begin{array}{c} \begin{array}{cccc} d & c & b & a \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \\ 1 & -3 & 9 & -27 \end{bmatrix} \end{array}$$

Q: Is A invertible?

Wolfram Alpha says "yes", so

$$\vec{v} = A^{-1} \cdot \vec{b}$$

$$= \frac{1}{24}\begin{bmatrix} -6 \\ -109 \\ 54 \\ 13 \end{bmatrix} = \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$$

The interpolating polynomial is

$$\boxed{y = \frac{13}{24}x^3 + \frac{54}{24}x^2 - \frac{109}{24}x - \frac{6}{24}}$$

Checked!

Proofs by Contradiction

Prove a mathematical statement by assuming the negation of what you're trying to prove. Reason from this assumption to something you know to be false; this is called a contradiction. You then know that, since the assumption yields false information, your original conclusion must be true.

Example 4: (uniqueness of inverse)

Show that inverses, if they exist, are unique.

Solution: Let A, B, C be $n \times n$ matrices with

A is invertible, with two inverses
$$\begin{cases} A \cdot B = B \cdot A = I_n \\ A \cdot C = C \cdot A = I_n \end{cases}$$

Suppose that $B \neq C$, i.e., we assume that inverses are not unique.

Then

$$B = B \cdot I_n = B \cdot (A \cdot C)$$ we assumed $A \cdot C = I_n$

$$= (B \cdot A) \cdot C$$ associativity of matrix multiplication

$$= (I_n) \cdot C$$ we assumed $B \cdot A = I_n$

$$= C$$

This contradicts our assumption that $B \neq C$.

So the assumption that $B \neq C$ must be in error, which implies that it is false. If it is false, its negation must be true, and so inverses are unique (if they exist).

Properties of Invertible Matrices

1) Products: Suppose A, B are $n \times n$ invertible matrices. Then $A \cdot B$ is also invertible

proof: (direct) I claim that the inverse of $A \cdot B$ is $B^{-1} \cdot A^{-1}$

Check: $(A \cdot B) \cdot (B^{-1} \cdot A^{-1})$

$= A \cdot (B \cdot B^{-1}) \cdot A^{-1}$ associativity

$= A \cdot (I_n) \cdot A^{-1}$ B invertible

$= (A \cdot I_n) \cdot A^{-1}$ associativity

$= A \cdot A^{-1} = I_n$ A invertible

This shows $AB \cdot (B^{-1} \cdot A^{-1}) = I_n$. A similar calculation shows

$(B^{-1} \cdot A^{-1}) \cdot (A \cdot B) = I_n$, so $B^{-1} \cdot A^{-1}$ is the inverse of $A \cdot B$. This is usually expressed as

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

2) Transposes

What is the transpose?

Given an $m \times n$ matrix A, the <u>transpose</u> of A, denoted by A^t, is the $n \times m$ matrix obtained by interchanging the rows of A with its columns.

Example 4: Let $A = \begin{bmatrix} -3 & 1 & 5 & 2 \\ 8 & 11 & 6 & 31 \end{bmatrix}$.

Find A^t.

Solution:

$$A^t = \begin{bmatrix} -3 & 8 \\ 1 & 11 \\ 5 & 6 \\ 2 & 31 \end{bmatrix}$$

1st row of A becomes 1st column of A^t

2nd row of A becomes 2nd column of A^t

An important property of transposes is that if A is an $m \times n$ matrix and B is an $n \times k$ matrix,

$$(A \cdot B)^t = B^t \cdot A^t$$

Using this fact, let's show: if A is an $n \times n$ invertible matrix, then the inverse of A^t is the transpose of A^{-1}.

Check: $A^t \cdot (A^{-1})^t = (A^{-1} \cdot A)^t$ $B = A^{-1}$

$= (I_n)^t$ A invertible

$= I_n$

Similarly, we can show

$$(A^{-1})^t \cdot A^t = I_n$$

This gets us that the inverse of A^t is the transpose of A^{-1}.

This is usually expressed as

$$(A^t)^{-1} = (A^{-1})^t$$

These facts are yours to use for the rest of the class whenever necessary.

Q: Is there a way to check whether a matrix is invertible?

A: Yes! Determinants.