Introduction to Matrices

Start with a system of linear equations.

Forget about the variables and record the coefficients and any other numbers in an array called a matrix.

The matrix will have m horizontal rows and n vertical columns. We say the matrix is "m by n"

and write $m \times n$

Example 1: Given the points $(-2,3)$, $(1,4)$, and $(5,-1)$, find the interpolating polynomial through these three points

Solution: Let $p(x) = ax^2+bx+c$. Plug in points.

$$3 = p(-2) = 4a - 2b + c$$

$$4 = p(1) = a + b + c$$

$$-1 = p(5) = 25a + 5b + c$$

Forget about the variables.

$$3 = p(-2) = 4a - 2b + c$$

$$4 = p(1) = a + b + c$$

$$-1 = p(5) = 25a + 5b + c$$

right-most column

left-most column

rows will be coefficients of a single equation

$$\begin{array}{c} \begin{array}{cccc} c & b & a & y\text{-values} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -2 & 4 & 3 \\ 1 & 5 & 25 & -1 \end{array}\right] \end{array}$$

Perform multiplying one equation and adding one to another just on the rows of numbers.

Goal:

$$\begin{array}{c} \begin{array}{ccccc} c & b & a & & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & s \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & r \end{array}\right] \end{array}$$

read as "$c = s$, $b = t$, $a = r$"

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & -2 & 4 & 3 \\ -1 & -1 & -1 & -4 \\ 1 & 5 & 25 & -1 \\ -1 & -1 & -1 & -4 \end{bmatrix}$$

1) Subtract Row 1 from Row 2 and Row 3

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -3 & 3 & -1 \\ 0 & 4 & 24 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -3 & 3 & -1 \\ 0 & 4 & 24 & -5 \end{bmatrix}$$

2) Divide Row 2 by -3

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & 1/3 \\ 0 & 4 & 24 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 4\cdot 1 & 4\cdot(-1) & 4(1/3) \\ 0 & 4 & 24 & -5 \end{bmatrix}$$

3) Subtract Row 2 from Row 1, subtract $4\cdot(\text{Row } 2)$ from Row 3.

$$\begin{bmatrix} 1 & 0 & 2 & 11/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 28 & -19/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 11/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 28 & -19/3 \end{bmatrix}$$

4) Divide Row 3 by 28

$$\begin{bmatrix} 1 & 0 & 2 & 11/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 1 & -19/84 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 11/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 1 & -19/84 \end{bmatrix}$$

5) Add Row 3 to Row 2, Subtract $2 \cdot$ (Row 3) from Row 1

$$\begin{array}{c} \begin{array}{cccc} c & b & a & \text{Solutions} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 346/84 \\ 0 & 1 & 0 & 9/84 \\ 0 & 0 & 1 & -19/84 \end{array}\right] \end{array}$$

We have

$$c = \frac{346}{84} = \frac{173}{42}$$

$$b = 9/84 = 3/28$$

$$a = -19/84$$

quadratic : $\boxed{p(x) = -\frac{19}{84}x^2 + \frac{3}{28}x + \frac{173}{42}}$

Works!

Coefficient and Augmented Matrices

Given a system of linear equations, the coefficient matrix C is the matrix of all coefficients of variables. The augmented matrix A is the matrix

$$A = \begin{array}{c} \text{coefficient matrix} \\ \left[\begin{array}{c|c} C & y \end{array}\right] \end{array}$$

where y represents the values the equations are equal to

Row Echelon Form

A matrix is in Row Echelon Form if

1) All rows consisting of all zeros occur at the bottom of the matrix

2) The first nonzero entry from the left in any row is equal to 1 (a leading 1)

3) Each leading one is to the right of the leading ones in the rows above it

Reduced Row Echelon Form

(RREF)

A matrix is in RREF if it is in Row Echelon Form and every entry in a column containing a leading one is zero (except for the leading one).

Goal: find solutions to linear systems by obtaining an augmented matrix and then putting it in RREF by adding multiples of rows to eachother (row reduction)

Given a matrix, it has a <u>unique</u> reduced row echelon form, but row echelon forms need not be unique.

Example 2:

Which of these matrices is in RREF?

a) $\begin{bmatrix} 1 & 3 & -6 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

a) not in RREF because of the -6 above the leading one, it should be zero.

b) in RREF

c) not in RREF, we need to interchange the rows to get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Systems of Linear Equations and RREF

Given a system of linear equations and the associated augmented matrix, we put it in RREF. There are 3 possible outcomes:

1) you see a row of all zeros, ending in a one = no solution for the system.

2) Every nonzero row has a leading 1 and a number to the very right and we are not in situation 1) = unique solution for the system.

3) Anything else = infinitely many solutions

Example 3: How many solutions do the systems of linear equations, represented by the following augmented matrices in RREF, have?

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 8 & 6 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \end{bmatrix}$

Solutions:

a) no solution, 3^{rd} row is a row of zeros ending in a one (says "$0 = 1$", not true)

b) infinitely many solutions, it does not have a row of zeros ending in a one, and the rows do not all have a single one and a solution in the far right.

c) unique solution. If we represent our variables as a and b, we have

$$\begin{array}{c} \begin{array}{ccc} a & b & \text{solutions} \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -6 \end{array}\right] \end{array}$$

says $a = 3$, $b = -6$

Example 4: What are the dimensions of the following matrices?

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 8 & 6 \end{bmatrix}$

c) $\begin{bmatrix} 5 & 12 & -52 & 101.1 & 0 \\ 1 & -8 & 3.5 & 3/5 & 0 \\ 6 & 0 & \pi & 42 & 0 \\ 8 & 13 & \sqrt{2} & -23 & 0 \end{bmatrix}$

Solutions:

a) 3×3

rows (horizontal)　　columns (vertical)

b) 2×4　　(not 4×2!)

c) 4×5　　(not 5×4)