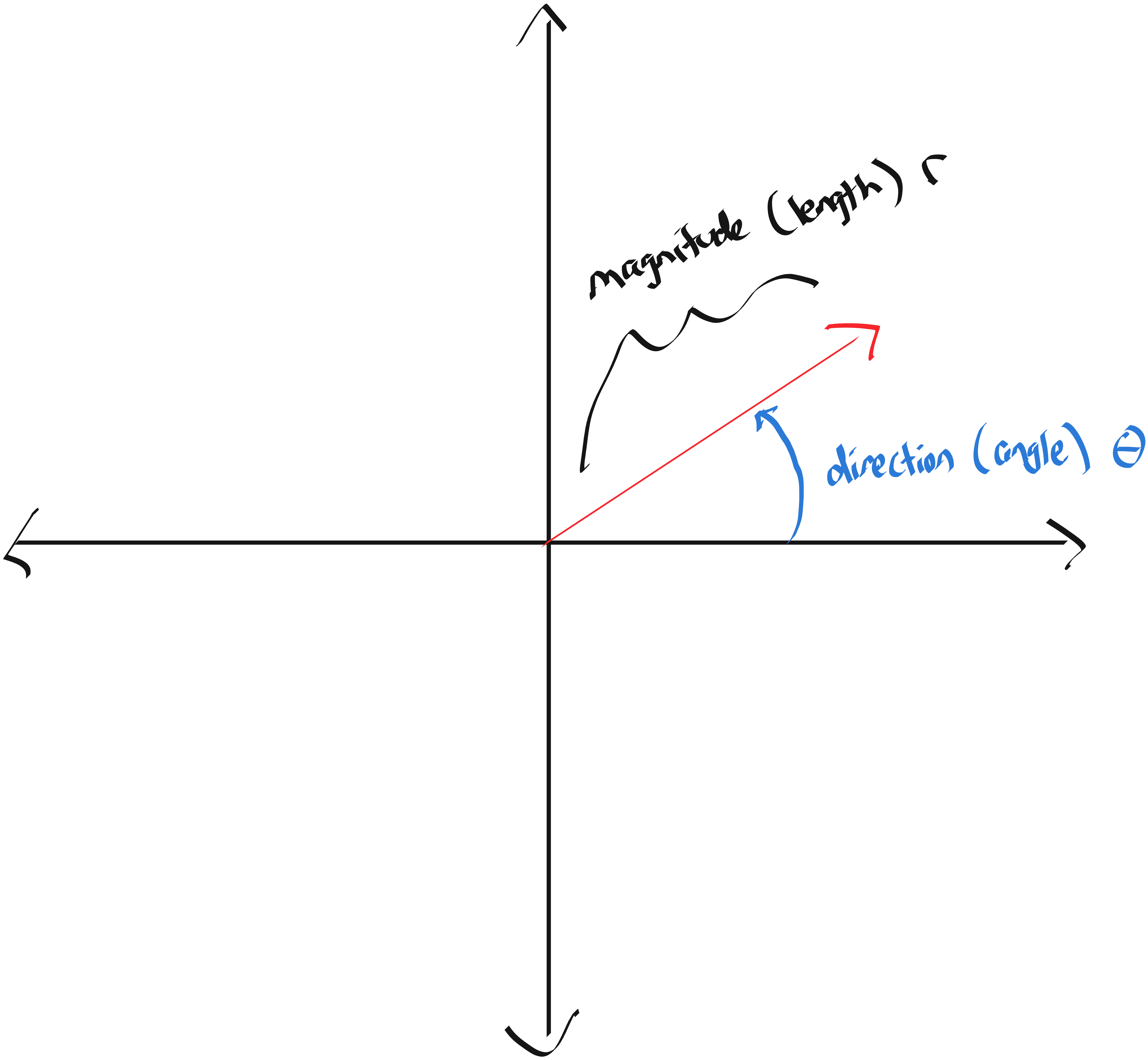
Vectors in $\mathbb{R}^n$

(Section 4.1)

Geometric intuition: a vector is a quantity possessing both magnitude (length) and direction (angle). Here, we're thinking about 2-dimensional vectors.

Picture



Example 1: Let $\vec{v}$ be the vector with magnitude 7 and angle $-\frac{2\pi}{3}$. Find the Cartesian representation of v.

Solution: Our vectors start at the origin and terminate at a point (x,y). We want to find x and y.

From Calc II: if r is the magnitude and θ is the direction,

$$\boxed{\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}}$$

Using the formulas,

$$r = 7$$

$$\theta = -\frac{2\pi}{3}, \quad \text{so}$$

$$x = r\cos(\theta) = 7\cos\left(-\frac{2\pi}{3}\right)$$

$$= 7\cos\left(\frac{2\pi}{3}\right)$$

cosine is even

$$= -\frac{7}{2}$$

$$y = r\sin(\theta) = 7\sin\left(-\frac{2\pi}{3}\right)$$

$$= -7\sin\left(\frac{2\pi}{3}\right)$$

sine is odd

$$= -\frac{7\sqrt{3}}{2}$$

The coordinates are written as

$$\left\langle -\frac{7}{2}, -\frac{7\sqrt{3}}{2} \right\rangle$$

$\mathbb{R}^n$

$\mathbb{R}$ = the real numbers (all fractions of numbers a/b where a, b are counting numbers or $a = 0$, and negatives of such numbers, plus weird stuff like $\sqrt{2}$ or π). Not imaginary numbers!

$\mathbb{R}^n$ denotes all ordered n-tuples of real numbers.

$n = 1, 2, 3, 4, \ldots$ (counting numbers)

- $\mathbb{R}^1 = \mathbb{R}$
- $\mathbb{R}^2 = (x, y)$ where x, y are in $\mathbb{R}$

we will sometimes write these as

a row $\begin{bmatrix} x & y \end{bmatrix}$ or a column

$\begin{bmatrix} x \\ y \end{bmatrix}$ when we're talking about

vectors in $\mathbb{R}^2$.

- $\mathbb{R}^3 = (x, y, z)$ where x, y, z are in $\mathbb{R}$.

Sometimes we write $\begin{bmatrix} x & y & z \end{bmatrix}$ or

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for vectors in $\mathbb{R}^3$

In $\mathbb{R}^3$, vectors are specified by their length and two angles.

- $\mathbb{R}^n = (x_1, x_2, \dots, x_n)$ where

  $x_1, x_2, \dots, x_n$ are all in $\mathbb{R}$.

  This is a finite list of real numbers, not an infinite sequence.

For vectors in $\mathbb{R}^n$:

$$[x_1 \ x_2 \ \dots \ x_n] \text{ or } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

"Ordered" means that switching the order of numbers doesn't necessarily give you the same vector; for example,

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Convention: All vectors emanate from the origin.

Vector Addition

Given vectors $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$,

we add the vectors by

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}}$$

= adding corresponding entries

Warning! You can only add vectors of the same size.

For example

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$ is OK, but

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 15 \end{bmatrix}$$ is not OK

Scalar Multiplication

Given $c \in \mathbb{R}$ and $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$.

We scalar multiply as follows:

$$c \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

= multiply every entry by c

Example 2: Given vectors

$$\vec{v} = \begin{bmatrix} 6 \\ 8 \\ -4 \end{bmatrix} \quad \text{and}$$

$$\vec{w} = \begin{bmatrix} 10 \\ 11 \\ 7 \end{bmatrix}, \quad \text{compute}$$

$$\frac{3}{2}\vec{v} + 5\vec{w}.$$

Solution:

$$\frac{3}{2}\vec{v} = \begin{bmatrix} 3/2 \cdot 6 \\ 3/2 \cdot 8 \\ 3/2 \cdot (-4) \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ -6 \end{bmatrix}$$

$$5\vec{w} = \begin{bmatrix} 5 \cdot 10 \\ 5 \cdot 11 \\ 5 \cdot 7 \end{bmatrix} = \begin{bmatrix} 50 \\ 55 \\ 35 \end{bmatrix}$$

so

$$\frac{3}{2}\vec{v} + 5\vec{w} = \begin{bmatrix} 9 \\ 12 \\ -6 \end{bmatrix} + \begin{bmatrix} 50 \\ 55 \\ 35 \end{bmatrix}$$

$$= \begin{bmatrix} 59 \\ 67 \\ 29 \end{bmatrix}$$

Linear Combinations

Take a collection of vectors S in $\mathbb{R}^n$.

A vector $\vec{v}$ in $\mathbb{R}^n$ is a **linear combination** of vectors in S if there is a counting number k and vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ in S with

$$\boxed{\vec{v} = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_k\vec{w}_k}$$

for some scalars $c_1, c_2, \ldots, c_k$

i.e. you can obtain $\vec{v}$ by scalor multiplying or adding ("or" means potentially both) vectors in S.

How to check if one vector is a linear combination of given vectors:

Write down your vector equation, turn it into a system of linear equations, then solve the equations (maybe using matrices).

Example 3: Is $\begin{bmatrix} 5 \\ 11 \\ 12 \\ 8 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 8 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 7 \\ -10 \\ 5 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} -18 \\ 24 \\ 0 \\ 6 \end{bmatrix}$?

Solution: What we want to know is whether there are scalars $c_1, c_2,$ and c_3 with

$$c_1 \begin{bmatrix} 1 \\ 8 \\ -1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ -10 \\ 5 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} -18 \\ 24 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 12 \\ 8 \end{bmatrix}$$

$$\underbrace{c_1 \begin{bmatrix} 1 \\ 8 \\ -1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ -10 \\ 5 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} -18 \\ 24 \\ 0 \\ 6 \end{bmatrix}} = \begin{bmatrix} 5 \\ 11 \\ 12 \\ 8 \end{bmatrix}$$

write out

$$c_1 \begin{bmatrix} 1 \\ 8 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 8c_1 \\ -c_1 \\ 3c_1 \end{bmatrix}$$

$$c_2 \begin{bmatrix} 7 \\ -10 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 7c_2 \\ -10c_2 \\ 5c_2 \\ -2c_2 \end{bmatrix}$$

$$c_3 \begin{bmatrix} -18 \\ 24 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} -18c_3 \\ 24c_3 \\ 0 \\ 6c_3 \end{bmatrix}$$

add all these vectors

$$\begin{bmatrix} c_1 \\ 8c_1 \\ -c_1 \\ 3c_1 \end{bmatrix} + \begin{bmatrix} 7c_2 \\ -10c_2 \\ 5c_2 \\ -2c_2 \end{bmatrix} + \begin{bmatrix} -18c_3 \\ 24c_3 \\ 0 \\ 6c_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + 7c_2 - 18c_3 \\ 8c_1 - 10c_2 + 24c_3 \\ -c_1 + 5c_2 \\ 3c_1 - 2c_2 + 6c_3 \end{bmatrix}$$

Set equal to $\begin{bmatrix} 5 \\ 11 \\ 12 \\ 8 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} c_1 + 7c_2 - 18c_3 \\ 8c_1 - 10c_2 + 24c_3 \\ -c_1 + 5c_2 \\ 3c_1 - 2c_2 + 6c_3 \end{bmatrix}}_{\text{vector in } \mathbb{R}^4} = \underbrace{\begin{bmatrix} 5 \\ 11 \\ 12 \\ 8 \end{bmatrix}}_{\text{vector in } \mathbb{R}^4}$$

If these vectors are equal, their corresponding entries must be equal, so

$c_1 + 7c_2 - 18c_3 = 5$ 1st entry

$8c_1 - 10c_2 + 24c_3 = 11$ 2nd "

$-c_1 + 5c_2 = 12$ 3rd "

$3c_1 - 2c_2 + 6c_3 = 8$ 4th "

$$c_1 + 7c_2 - 18c_3 = 5$$

$$8c_1 - 10c_2 + 24c_3 = 11$$

$$-c_1 + 5c_2 = 12$$

$$3c_1 - 2c_2 + 6c_3 = 8$$

System of linear equations!

Solve using matrices

$$\begin{array}{c} \begin{array}{ccccc} c_1 & c_2 & c_3 & & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 7 & -18 & 5 \\ 8 & -10 & 24 & 11 \\ -1 & 5 & 0 & 12 \\ 3 & -2 & 6 & 8 \end{array}\right] \end{array}$$

row reduce

Wolfram Alpha

RREF (matrix)

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \text{Solutions} \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No Solution.

So no such c_1, c_2, c_3 exist.

Terminology: (span) Given a collection of vectors S in $\mathbb{R}^n$, the <u>span</u> of S, denoted by $\mathrm{span}(S)$, is the collection of all linear combinations of vectors in S. If $\mathrm{span}(S) = \mathbb{R}^n$, we say that S is a <u>spanning set</u> for $\mathbb{R}^n$, i.e., every vector in $\mathbb{R}^n$ is a linear combination of vectors in S.

Dot Products

Row to column: given vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, and write

$$\vec{v} = [v_1 \ v_2 \ \cdots \ v_n] \text{ and}$$

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$ for $v_1, v_2, \ldots, v_n$ and

$w_1, w_2, \ldots, w_n$ in $\mathbb{R}$.

The dot product of $\vec{v}$ with $\vec{w}$ is the real number given by

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

vectors

$$= v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

$$= \underbrace{\sum_{i=1}^{n} v_i w_i}_{\text{a real number}}$$

Example 4:

Let $\vec{v} = \begin{bmatrix} 2 & -6 \end{bmatrix}$ and

$$\vec{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Compute $\vec{v} \cdot \vec{w}$.

Solution:

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} 2 & -6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= (2 \cdot 3) + (-6 \cdot 4)$$

$$= \boxed{-18}$$

Remark: if you want to compute $\vec{w} \cdot \vec{v}$ in the previous example, write $\vec{w}$ as a row vector and $\vec{v}$ as a column vector:

$$\vec{w} \cdot \vec{v} = \begin{bmatrix} 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$= -18$$

In this manner, we can see that

$$\boxed{\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}}$$

Geometric Interpretations

1) Scalar multiplication: if $\vec{v}$ is a vector in $\mathbb{R}^n$ and c is a scalar, then $c \cdot \vec{v}$ is

a) the zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ if $c = 0$,

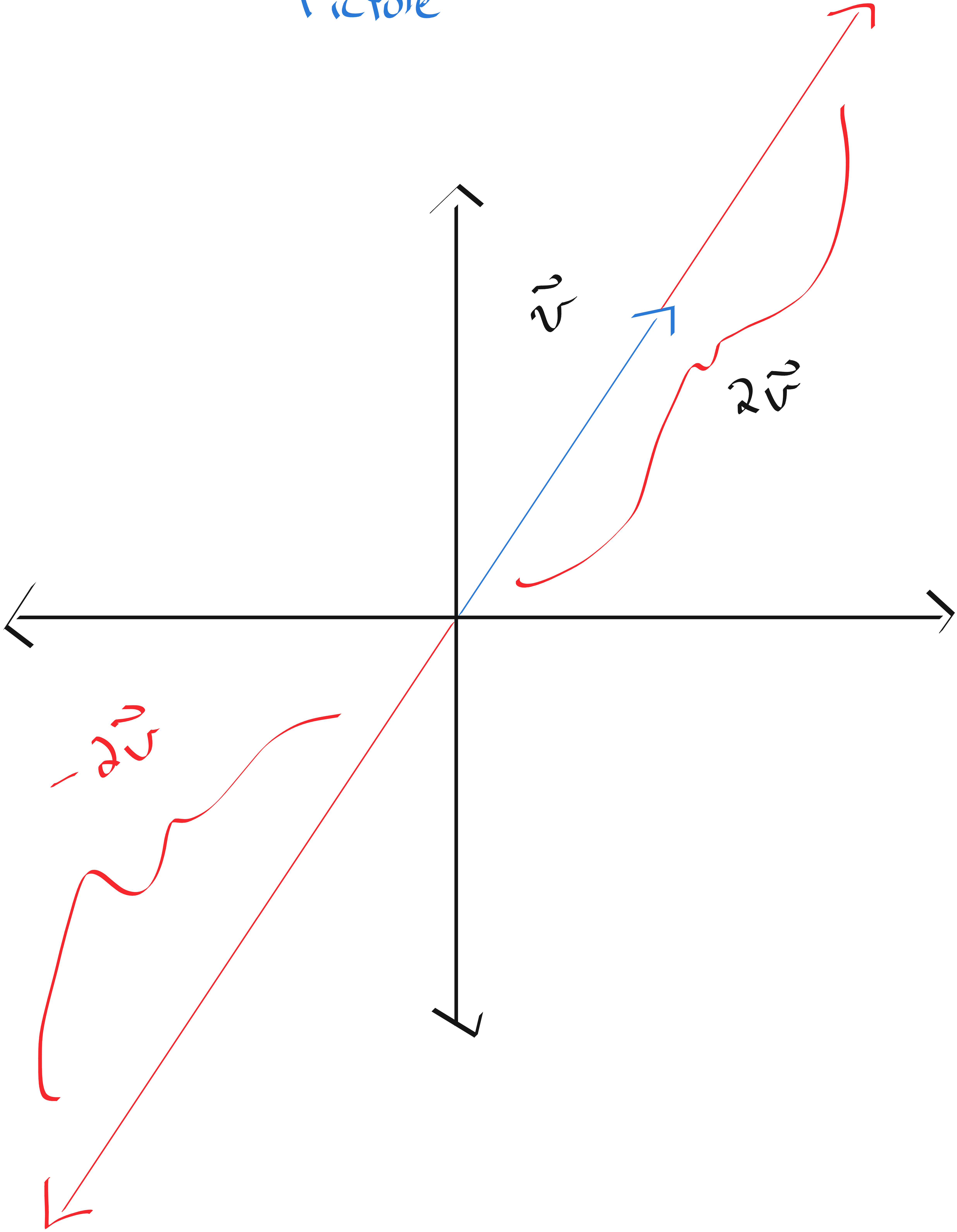
b) $\vec{v}$ if $c = 1$,

c) a stretched version of $\vec{v}$ (same direction, bigger magnitude) if $c > 1$,

d) a shrunken version of $\vec{v}$ (same direction, smaller magnitude) if $0 < c < 1$

e) a vector pointing in the opposite direction of $\vec{v}$, with magnitude governed by $|c|$, if $c < 0$.

Picture



2) Vector Addition: if $\vec{v}, \vec{w}$ are vectors in $\mathbb{R}^n$, the vector $\vec{v} + \vec{w}$ is computed geometrically by

a) drawing $\vec{v}$,

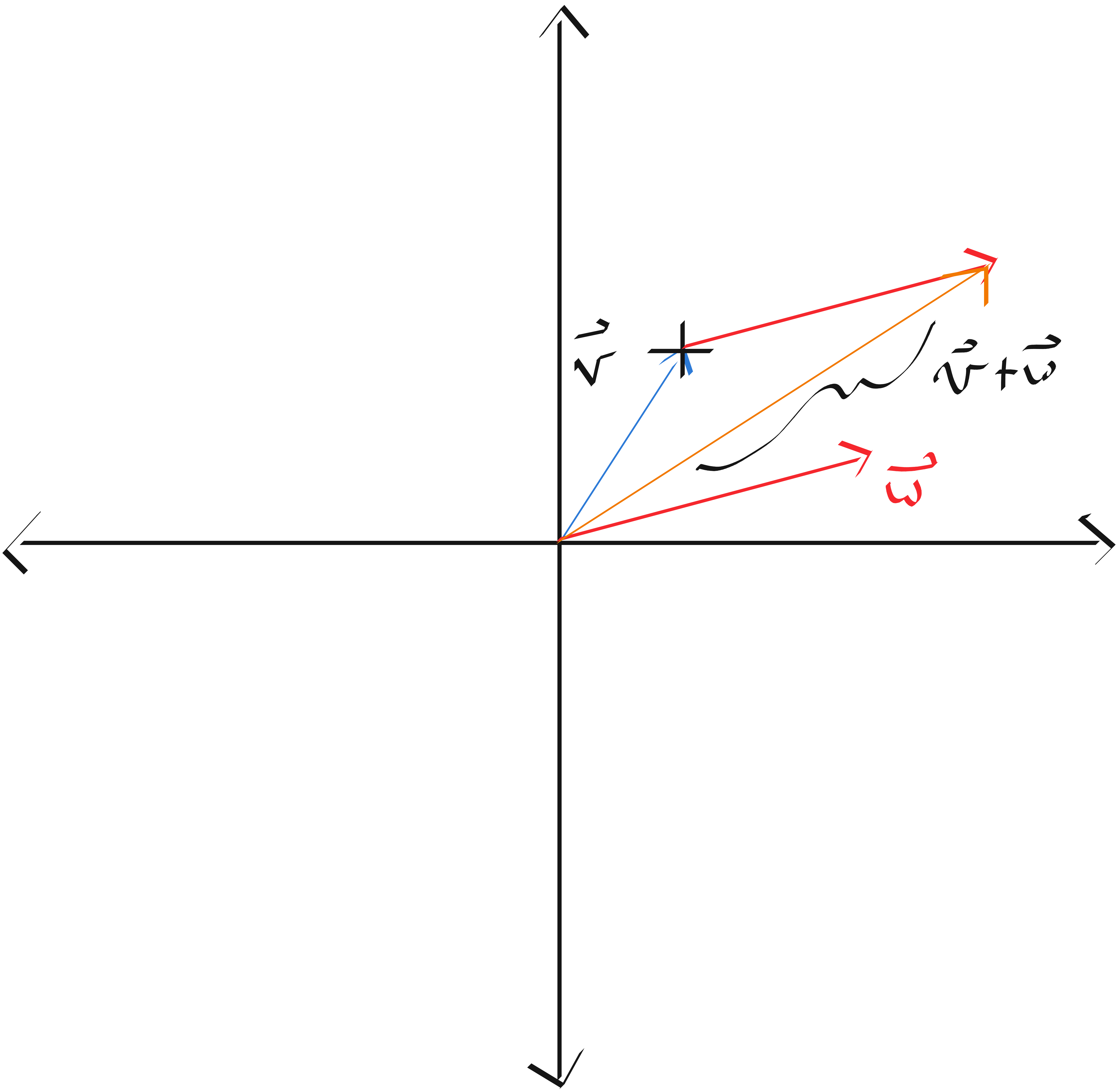
b) making a new coordinate axis where $\vec{v}$ terminates,

c) drawing $\vec{w}$ from the origin of this new axis,

d) connecting the original origin to where $\vec{w}$ terminates.

This gives the vector $\vec{v} + \vec{w}$

Picture



3) Span

The zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^n$

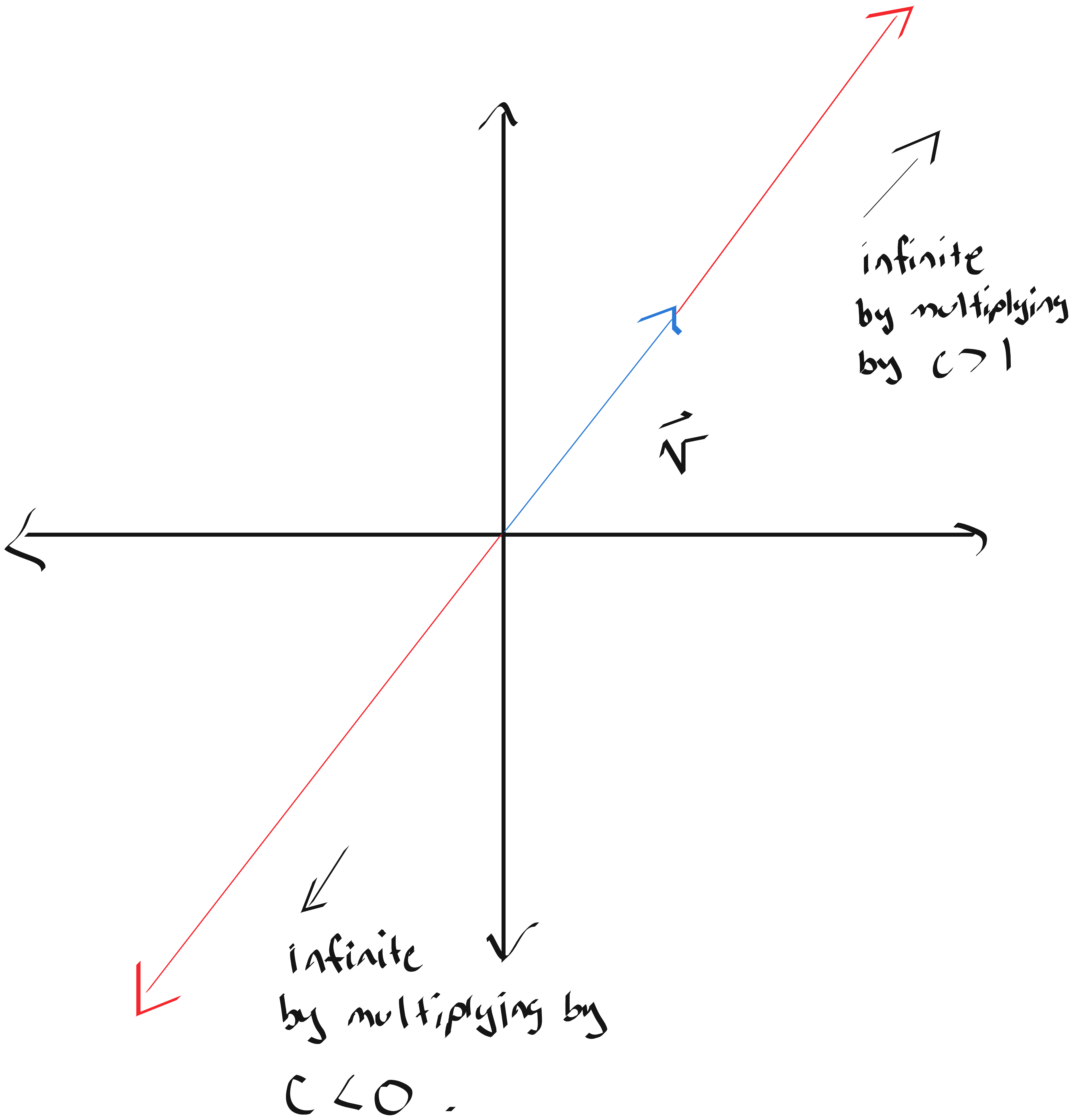
Adding the zero vector to itself yields the zero vector again. Similarly, multiplying the zero vector by any scalar also yields the zero vector. Denoting the zero vector by $\vec{0}$,

geometrically,

$$\boxed{\text{span}(\vec{0}) = \text{a point}}$$

One nonzero vector

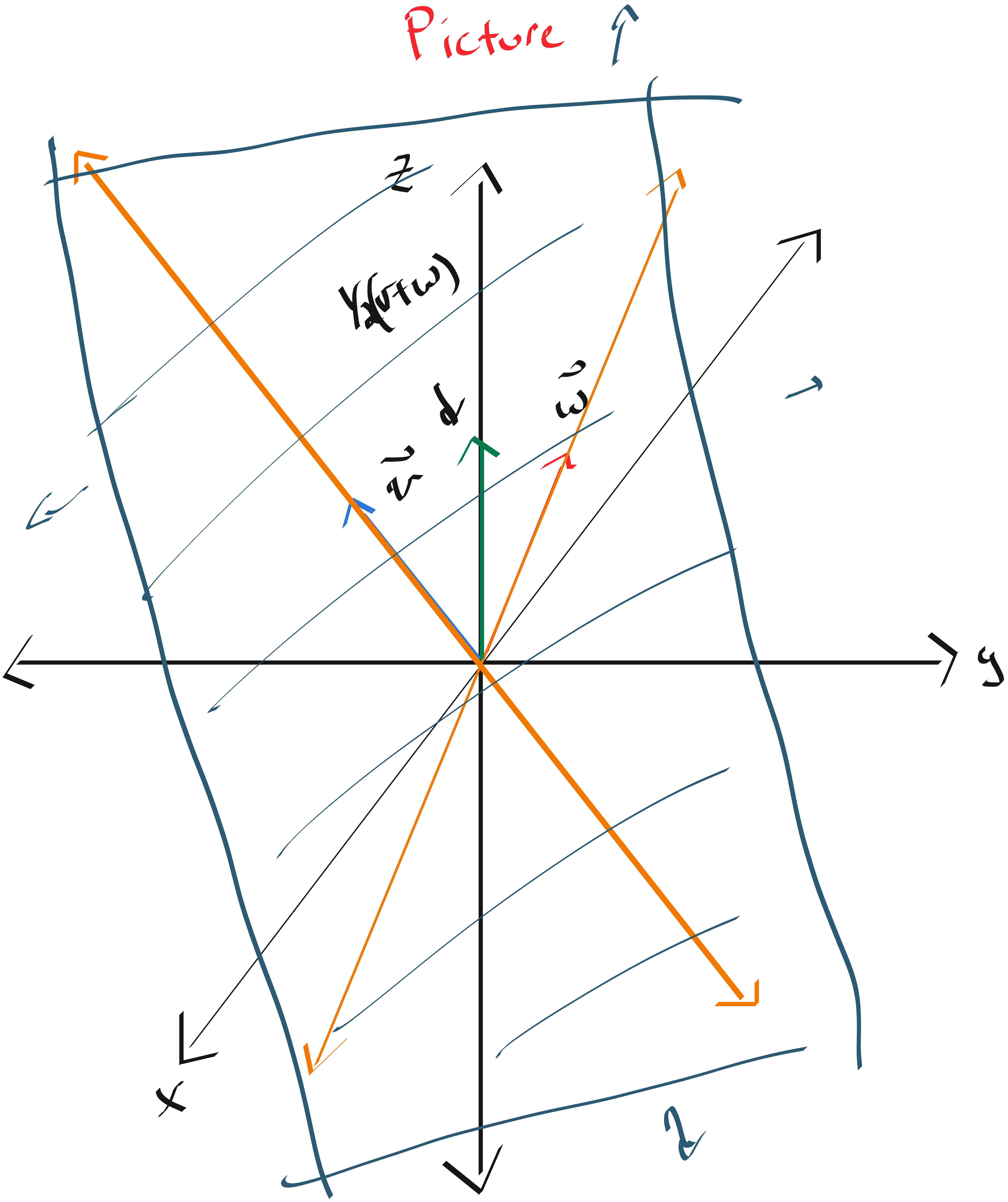
Picture



The span, or all linear combinations, of a single <u>nonzero</u> vector is a line, geometrically. We can get away with only scaler multiples of $\vec{v}$ in this case.

Two nonzero vectors

Picture



The span, or all linear combinations, of two nonzero vectors is a plane, geometrically, unless one vector is a scalar multiple of the other. In this case, we get a line again.

Three nonzero vectors (?)

Case 1: all three vectors point in the same, or opposite, direction.

Then the span of these three vectors is a line.

Case 2: not all three vectors are on the same line, but one is in the span of the other two.

Then the span of these three vectors is a plane.

<u>Case 3:</u> not all three vectors are on a line and not all three vectors are in a plane. Then the span of these three vectors is all of three-dimensional space.

Important observations on Span

1) The zero vector in $\mathbb{R}^n$ is in the span of any collection of vectors. Just make scalars equal zero.

2) Any vector is a linear combination of itself, make the scalar equal to one for that vector and don't add any other vectors.

3) If you start with nonzero vectors, the span is always an infinite collection of vectors; just choose all scalar multiples of a single vector.

4) Dot Product - later

If the two vectors are equal

If not . . .

Reduction :

Further reduction

Even further reduction

Example 5 : (angle between vectors)

Example 6: