Math 412/512 Assignment 1

Due Thursday, September 19

1) Use mathematical induction to complete part a) and, if you wish, part b).

a) Let S be a finite set with n elements and let T be a set with $n+1$ elements. If $f : T \to S$ is a function, prove that f is not an injection.

b) Verify that if S is finite, then any injection from S to itself is also surjective.

c) Give an example of an injection $f : \mathbb{R} \to \mathbb{R}$ that is not surjective.

2) This problem will test both your ability to construct a reasonable argument and your compliance with definitions: Is $\mathbb{R}$ a vector space over $\mathbb{Q}$? Prove or disprove.

3) Let V be the vector space over $\mathbb{R}$ consisting of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Equip V with the multiplication $(f \cdot g)(x) = f(x) \cdot g(x)$ for $f, g \in V$ and all $x \in \mathbb{R}$. If addition is given by the vector space structure, does the multiplication make V into a field? Either prove or explicitly demonstrate which of the field axioms fail.

4) (#21, Section 1.3) Show that the set of convergent sequences $\{a_n\}_{n\in\mathbb{N}}$ (i.e., those for which $\lim_{n\to\infty} a_n$ exists) is a subspace of the vector space over $\mathbb{R}$ of all sequences of real numbers.

5) The space $\ell_\infty(\mathbb{N})$ is the vector space of all complex sequences that are *bounded*; that is, a sequence $\{a_n\}_{n\in\mathbb{N}}$ is in $\ell_\infty(\mathbb{N})$ if and only if there exists a non-negative real number M with $|a_n| \leq M$ for all $n \in \mathbb{N}$.

a) Give an example of a sequence of complex numbers that is in $\ell_\infty(\mathbb{N})$ and a sequence of complex numbers that is not in $\ell_\infty(\mathbb{N})$.

b) Does the space remain unchanged if we replace "$|a_n| \leq M$" with "$|a_n| < M$" in the definition?

c) Prove that $\ell_\infty(\mathbb{N})$ is a subspace of the vector space over $\mathbb{C}$ of all sequences of *complex* numbers (you may assume that the space of all sequences of complex numbers is a vector space).