Math 412/512 Assignment 2

Due Monday, September 30

1) Let $\mathbb{F}$ be a field and let V be a vector space over $\mathbb{F}$. Show that for all $w \in V$, $(-1_{\mathbb{F}}) \cdot w = -w$ where $1_{\mathbb{F}}$ is the multiplicative identity of $\mathbb{F}$.

2) (#20, Section 1.3) Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for any scalars $a_1, a_2, \ldots, a_n$. Conclude that $span(W) = W$.

3) (#13, Section 1.4) Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $span(S_1) \subseteq span(S_2)$. In particular, if $S_1 \subseteq S_2$ and $span(S_1) = V$, deduce that $span(S_2) = V$.

4) (#9, Section 1.5) Let u and v be distinct elements in a vector space V. Prove that $\{u, v\} \subset V$ is linearly dependent if and only if u or v is a multiple of the other.

5) Let $C(\mathbb{R})$ denote the vector space over $\mathbb{R}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Fix $x \in \mathbb{R}$.

a) Let

$$W_x = \{f \in C(\mathbb{R}) \mid f(x) = 0\}.$$

For example, $W_{\sqrt{2}} = \{f \in C(\mathbb{R}) \mid f(\sqrt{2}) = 0\}$. Show that W_x is a vector subspace of $C(\mathbb{R})$ for all $x \in \mathbb{R}$.

b) Choose $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and let $W_{x,\alpha} = \{f \in C(\mathbb{R}) \mid f(x) = \alpha\}$. Is $W_{x,\alpha}$ a subspace of $C(\mathbb{R})$? Either prove or provide reasons for why not.

6) a) Let c_{00} be the subspace of all sequences of complex numbers that are "eventually zero." More precisely, a sequence of complex numbers $x = (x_i)_{i=1}^{\infty}$ is in c_{00} if and only if there is a natural number N so that $x_n = 0$ for all $n \geq N$. Consider c_{00} as a vector space over $\mathbb{C}$.

Let $\{e_i\}_{i \in \mathbb{N}}$ be the set where, for each $i \in \mathbb{N}$, e_i is the sequence in c_{00} given by

$$(e_i)_n = \begin{cases} 1 & n = i \\ 0 & n \neq i. \end{cases}$$

So e_1 is the sequence with a 1 in the first entry and zeros in all other entries, e_2 is the sequence with a 1 in the second entry and zeros in all other entries, etc. Show that $\{e_i\}_{i\in\mathbb{N}}$ is a basis for c_{00}.

b) Recall the definition of $\ell_\infty(\mathbb{N})$ (as a vector space over $\mathbb{C}$) from the previous homework. If $\{e_i\}_{i\in\mathbb{N}}$ is the set defined in part a) of this problem, is $\{e_i\}_{i\in\mathbb{N}}$ either a linearly independent or spanning subset of $\ell_\infty(\mathbb{N})$? Prove that your assertion is correct.

Extra Credit: I will accept no written solutions. You must explain your proof to me in my office.

For a field F and a vector space V over F, the *Grassmannian* $Gr(k, V)$ is the collection of all k-dimensional linear subspaces of V. The *order* of $Gr(k, V)$ is the number of distinct k-dimensional linear subspaces of V.

a) Determine, with proof, the number of distinct one-dimensional subspaces of $\mathbb{Z}_p^n$, i.e., find the order of $Gr(1, \mathbb{Z}_p^n)$.

b) The cardinality of $Gr(k, \mathbb{Z}_p^n)$ for $k \leq n$ is given by the formula $\dfrac{[n]_p!}{[n-k]_p![k]_p!}$ where for $m \in \mathbb{N}$,

$$[m]_p = \frac{p^m - 1}{p - 1}$$

and $[m]_p!$ is defined by

$$[m]_p! = [m]_p \cdot [m-1]_p \cdots [2]_p \cdot [1]_p.$$

Prove this formula for $1 < k < n$.

After you're done with this, in your own time, you should compute $\dfrac{[n]_p!}{[n-k]_p![k]_p!}$ for arbitrary p, n, and k, then take the limit as $p \to 1$. Ruminate on your answer and you may start to get some idea of what people mean by "the field with one element."