Math 413/513 Assignment 3

Due Monday, October 14

1) (#20, Section 1.6) Let V be a vector space having dimension n, and let S be a subset of V that generates V.

a) Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite.)

b) Prove that S contains at least n vectors.

2) (#12, Section 6.1) Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set in V, and let $\{a_1, a_2, \ldots, a_k\}$ be scalars. Prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i^2| \|v_i\|^2.$$

3) Consider the vector space V over $\mathbb{R}$ of all polynomials with real coefficients. Let W denote the subset of all polynomials with only even powers in their expression, i.e., $p(x) \in W$ if and only if $\exists\, k \in \mathbb{N}$ and scalars $a_0, a_1, a_2, \ldots, a_k$ such that

$$p(x) = \sum_{n=0}^{k} a_n x^{2n}.$$

Show that W is a subspace of V and find its dimension.

4) Prove the triangle inequality for complex numbers. NOTE: you are NOT allowed to assume any results on inner products to establish this.

5) In $C(\mathbb{R})$, the vector space over $\mathbb{R}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, let

$$W = \left\{ f \in C(\mathbb{R}) \middle| \int_{-\infty}^{\infty} |f(x)|\, dx < \infty \right\},$$

where the integral used is the ordinary (improper) Riemann integral. Define, for $f \in W$,

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

Show that $\|\cdot\|_1$ gives a norm on W.

6) The Schroder-Bernstein Theorem: Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Note that if $y \in f(X)$, i.e., $y \in ran(f)$, there exists a *unique* $x \in X$ such that $f(x) = y$.

If we define $f^{-1}(y) = x$, then f^{-1} is a 1-1 function from $f(X)$ onto X. In a similar way, we can also define the 1-1 function $g^{-1} : g(Y) \to Y$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$.

a) Let $x \in X$ be arbitrary. Let the *chain* C_x be the set consisting of all elements of the form

$$\dots, f^{-1}(g^{-1}(x)), g^{-1}(x), x, f(x), g(f(x)), f(g(f(x))), \dots \tag{1}$$

Explain why the (distinct) number of elements to the left of x in the above chain may be zero, finite, or infinite.

b) FOR JUST THIS PART ONLY, let $X = \mathbb{Z}$ and $Y = 2\mathbb{Z}$, the even integers. Define $f : \mathbb{Z} \to 2\mathbb{Z}$, $f(n) = 4n$ and $g : 2\mathbb{Z} \to \mathbb{Z}$, $g(m) = m + 3$. Calculate the left (inverses) part of a chain starting with $x = 3$ and $x = 31$.

c) Show that any two chains are either identical or completely disjoint.

d) Note that the terms of the chain in (1) alternate between elements of X and elements of Y. Given a chain C_x, we want to focus on $C_x \cap Y$, which is just the part of the chain that sits in Y.

Define the set A to be the union of all chains C_x satisfying $C_x \cap Y \subseteq f(X)$. Let B consist of the union of the remaining chains not in A. Show that any chain contained in B must be of the form

$$y, g(y), f(g(y)), g(f(g(y))), \dots$$

where y is an element of Y that is *not* in $f(X)$.

e) Let $X_1 = A \cap X$, $X_2 = B \cap X$, $Y_1 = A \cap Y$, and $Y_2 = B \cap Y$. Show that f maps X_1 *onto* Y_1 and that g maps Y_2 onto X_2. Use this information to show that X and Y have the same cardinality.

Extra Credit: Two norms $\|\cdot\|$ and $|||\cdot|||$ on a vector space V over $\mathbb{R}$ or $\mathbb{C}$ are said to be *equivalent* if there are positive constants C and D such that, for all $x \in V$,

$$C\|x\| \leq |||x||| \leq D\|x\|.$$

Show that all p-norms on $\mathbb{R}^n$ are equivalent for $1 \leq p \leq \infty$. You may also want to try to prove the harder fact that *any* two norms on $\mathbb{R}^n$ are equivalent.