**Math 413/513 Assignment 4**

**Due Tuesday, November 12**

**1)** For example a), prove that $T$ is a linear transformation, find bases for $N(T)$ and $R(T)$, compute the nullity and rank, then determine whether $T$ is one-to-one or onto. For example b), assume that $T$ is linear and then find bases for $N(T)$ and $R(T)$, compute the nullity and rank, and determine whether $T$ is one-to-one or onto. Your answers for b) should depend on $n$!

a) (#3, Section 2.1) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

b) (#6, Section 2.1) $T : M_n(\mathbb{F}) \to \mathbb{F}$ defined by $T(A) = \mathrm{tr}(A)$ (see the book or the 10/3 notes for the definition of tr).

**2)** Let $V$ be a vector space over $\mathbb{F}$ and let $T : V \to V$ be linear. A subspace $W$ of $V$ is said to be *$T$-invariant* if $T(W) \subseteq W$.

a) (#28, Section 2.1) Prove that the subspaces $\{0_V\}$, $V$, $R(T)$, and $N(T)$ are all $T$-invariant.

b) A $T$-invariant subspace $W$ is said to be *nontrivial* if $W \neq \{0_V\}$ and $W \neq V$. Are the results from a) sufficient to show that every linear operator $T$ from $V$ to $V$ has a nontrivial $T$-invariant subspace? Why or why not?

c) Consider $\mathbb{R}^2$ as a vector space over $\mathbb{R}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T((x, y)) = (y, x)$. Find two distinct, nontrivial $T$-invariant subspaces. Check that your answers are correct.

d) Consider $C(\mathbb{R})$ as a vector space over $\mathbb{R}$. For $f, g \in C(\mathbb{R})$, define the linear map $T_f : C(\mathbb{R}) \to C(\mathbb{R})$ by

$$T_f(g) = g \circ f.$$

So $(T_f(g))(x) = g(f(x))$ for all $x \in \mathbb{R}$. For every such $f$, find a nontrivial $T_f$-invariant subspace. Check that your answer is correct.

**3)** (#5, Section 2.4) Let $A$ be invertible. Prove that $A^t$ is invertible and $(A^t)^{-1} = (A^{-1})^t$.

**4)** We know from results in class that $M_2(\mathbb{R})$ and $\mathbb{R}^4$ are isomorphic as real vector spaces. Now consider the explicit isomorphism $T$ induced by

$$e_1 \mapsto e_{1,1},\ e_2 \mapsto e_{2,1},\ e_3 \mapsto e_{1,2},\ e_4 \mapsto e_{2,2}.$$

We can define a linear map $S_A$ from $M_2(\mathbb{R})$ to itself by taking $A, B \in M_2(\mathbb{R})$ and setting

$$S_A(B) = AB.$$

Under the isomorphism $T$, $S_A$ becomes a linear map from $\mathbb{R}^4$ to $\mathbb{R}^4$, hence $S_A$ is represented by an element of $M_4(\mathbb{R})$.

a) Find the matrix of $S_A$ under this isomorphism, with respect to the standard basis, and check that your answer is correct.

b) Find an explicit form for all matrices in $M_4(\mathbb{R})$ that commute with the image of every $S_A$ under $T$ and show that the set of all such matrices is also isomorphic to $\mathbb{R}^4$ as a real vector space.

**5)** Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$ and let $a \in \mathbb{R}$, $a \neq 1$.

a) Show that $a\mathcal{B} = \{ay \mid y \in \mathcal{B}\}$ is a basis for $\mathbb{R}$ over $\mathbb{Q}$ for all $a \neq 0$.

b) For $x \in \mathbb{R}$ and $y \in \mathcal{B}$, we may define the function $q_y : \mathbb{R} \to \mathbb{Q}$ where

$$q_y(x) = \text{the coefficient of } y \text{ in the expansion of } x.$$

We can then define $f : \mathbb{R} \to \mathbb{Q}$ by

$$f(x) = \sum_{y \in \mathcal{B}} q_y(x).$$

Note the sum is well-defined since all but finitely many coefficients are zero. Considering $f$ as a map between vector spaces over $\mathbb{Q}$, prove that $f$ is linear.

**Extra Credit:**1) Determine whether $\ell_1(\mathbb{N})$ and $\ell_2(\mathbb{N})$ are isomorphic as real vector spaces.

2) Prove that, with $\mathcal{B}$ as in #5, there exists $y \in \mathcal{B}$, $ay \notin \mathcal{B}$.