Math 413/513 Assignment 5

Due Tuesday, November 19

1) (#10, Section 2.5) Prove that if A and B are similar $n \times n$ matrices, then tr(A)=tr(B). *Hint:* Use Exercise 13 of Section 2.3.

2) (#13, Section 2.5) Let V be a finite dimensional vector space over a field $\mathbb{F}$, and let $\beta = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from $\mathbb{F}$. Define

$$x'_j = \sum_{i=1}^{n} Q_{i,j} x_i \quad \text{for } 1 \leq j \leq n,$$

and set $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

3) a) Let $A \in M_n(\mathbb{C})$. Prove that if A is invertible, then A^{-1} is unique.

b) Given the matrix $A = \begin{bmatrix} 1 & 8 \\ 3 & 5 \\ 2 & 2 \end{bmatrix}$, find all 2×3 matrices $B \in M_{2\times 3}(\mathbb{R})$ with $BA = I_2$.

4) (#15, Section 4.3) Prove that if $A, B \in M_n(\mathbb{F})$ are similar, then $\det(A) = \det(B)$.

5) (#2, Section 5.1) For each of the following linear operators T on a vector space V and ordered bases β, compute $[T]_\beta$ and determine whether β is a basis consisting of eigenvectors for T.

a) $V = \mathbb{R}^2$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$.

b) $V = \mathbb{R}^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ -c \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$

6) A linear transformation $T : V \to V$ where V is a finite-dimensional inner product space is called *positive semi-definite* if

$$\langle Th, h\rangle \geq 0$$

for all $h \in V$.

a) Prove that if $A \in M_n(\mathbb{C})$ is positive semi-definite with respect to the usual inner product and λ is an eigenvalue for A, then $\lambda \geq 0$.

b) If $A \in M_2(\mathbb{C})$ and A^* denotes the conjugate transpose of A ($(A^*)_{i,j} = \overline{A_{j,i}}$ for all $1 \leq i, j \leq n$), prove that A^*A is positive semi-definite.