Math 413/513 Assignment 6

Due Tuesday, December 3

1) Find the characteristic polynomial and the eigenvalues of the following matrices over $\mathbb{C}$.

a) $A = \begin{bmatrix} -1 & 6 \\ 3 & 11 \end{bmatrix}$

b) $B = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$

2) Let $A \in M_n(\mathbb{C})$ and $\alpha \in \mathbb{C}$. Using the definition of the determinant given in class, prove that if B is the matrix obtained by multiplying a single row of A by α, then

$$\det(B) = \alpha \det(A).$$

3) (#14, Section 5.1) For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues). You may assume that $A \in M_n(\mathbb{C})$ or $M_n(\mathbb{R})$.

4) a) If A and B are two similar $n \times n$ matrices, prove that A and B have the same characteristic polynomial, and hence, the same eigenvalues.

b) Show that if $T : \mathbb{C}^n \to \mathbb{C}^n$ is a linear map, then the eigenvalues of $[T]_\beta$ and the eigenvalues of $[T]_\gamma$ are the same for any two ordered bases β, γ of $\mathbb{C}^n$. Conclude that the eigenvalues of a linear transformation are independent of the matrix form given by a choice of basis.

5) Let $A \in M_n(\mathbb{C})$ and suppose A is invertible. Note that zero is not an eigenvalue of A.

a) Prove that λ is an eigenvalue of A if and only if $1/\lambda$ is an eigenvalue of A^{-1}.

b) Prove that A is diagonalizable if and only if A^{-1} is diagonalizable.

6) Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in M_2(\mathbb{R})$. Define a map $H_A : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$H_A(B) = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}$$

for all $B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \in M_2(\mathbb{R})$.

a) Prove that H_A is a linear map.

b) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Find the eigenvalues of H_A.

c) **(Extra Credit)** Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and define

$$\|H_A\|_\infty = \max_{B \in M_n(\mathbb{R}), \|B\|=1} \|H_A(B)\|.$$

Prove that $\|H_A\|_\infty = \|A\| = (1 + \sqrt{5})/2$ where $\|A\|$ is the matrix norm defined in the 11-11 notes. The number $(1 + \sqrt{5})/2$ is the so-called *golden ratio*. I will not accept written solutions, you must present your solution to me in my office on the board.

7) Let A and B be self-adjoint matrices in $M_n(\mathbb{C})$.

a) By considering $n = 2$, show that AB need not be self-adjoint.

b) (#4, Section 6.4) Prove that AB is self-adjoint if and only if $AB = BA$.

8) Let u be a unitary matrix in $M_2(\mathbb{R})$.

a) Prove that if $\{b_1, b_2\}$ is an orthonormal basis of $\mathbb{R}^2$, then $u(b_2)$ is determined up to a negative sign by $u(b_1)$.

b) Determine the matrix in the standard basis $\{e_1, e_2\}$ of every such u.

9) A matrix $A \in M_n(\mathbb{F})$ is said to be *nilpotent* if there exists $k \in \mathbb{N}$ with $A^k = 0_{M_n(\mathbb{F})}$.

a) Prove that if $A \in M_n(\mathbb{C})$ is nilpotent, then $\lambda = 0$ is the only eigenvalue of A.

b) If $\mathbb{F} = \mathbb{C}$ and A is both normal and nilpotent, show that $A = 0_{M_n(\mathbb{C})}$.