Math 413/513 Assignment 2

Due Thursday, September 25

1) (Problem 2.6) Is it possible that vectors v_1, v_2, v_3 are linearly dependent, but the vectors $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, and $w_3 = v_3 + v_1$ are linearly *independent*?

2) Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $span(S_1) \subseteq span(S_2)$. In particular, if $S_1 \subseteq S_2$ and $span(S_1) = V$, then $span(S_2) = V$.

3) (Problem 3.4) Find 3×3 matrices representing the transformations of $\mathbb{R}^3$ which

a) project every vector onto the xy-plane;

b) reflect every vector through the xy-plane;

c) rotate the xy-plane through $30°$, leaving the z-axis alone.

4) (Problem 5.5) Find linear transformations $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ such that $AB = 0$ but $BA \neq 0$. (Be sure to show your answer is correct.)

5) Let $A \in M_2(\mathbb{R})$.

a) Suppose $\exists x \in \mathbb{R}^2$, $x \neq \vec{0}$, $Ax = \vec{0}$. Must there exist $y \in \mathbb{R}^2$, $y \neq \vec{0}$, such that $A^t y = \vec{0}$? Prove or exhibit a counterexample.

b) Let $x \in \mathbb{R}^2$, $x \neq \vec{0}$. Show that $Ax = \vec{0}$ if and only if $A^t Ax = \vec{0}$.

Extra Credit: I will accept no written solutions. You must explain your proof to me in my office.

Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$ and let $a \in \mathbb{R}$, $a \neq 1$.

a) (5 points) Show that $a\mathcal{B} = \{ay \mid y \in \mathcal{B}\}$ is a basis for $\mathbb{R}$ over $\mathbb{Q}$ for all $a \neq 0$.

b) (5 points) For $x \in \mathbb{R}$ and $y \in \mathcal{B}$, we may define the function $q_y : \mathbb{R} \to \mathbb{Q}$ where

$$q_y(x) = \text{the coefficient of } y \text{ in the expansion of } x.$$

We can then define $f : \mathbb{R} \to \mathbb{Q}$ by

$$f(x) = \sum_{y \in \mathcal{B}} q_y(x).$$

Note the sum is well-defined since all but finitely many coefficients are zero. Considering f as a map between vector spaces over $\mathbb{Q}$, prove that f is linear.

c) (many points) Prove that there exists $y \in \mathcal{B}$, $ay \notin \mathcal{B}$. Be warned: this problem is quite tricky!