Math 413/513 Assignment 3

Due Tuesday, October 7

1) Tex up what was assigned to you.

2) (Exercise 5.6) Prove Theorem 5.1, i.e. prove that trace(AB)=trace(BA).

3) a) (Exercise 7.1) Let X and Y be subspaces of a vector space V. Prove that $X \cap Y$ is a subspace of V.

b) (Exercise 7.4) Let X and Y be subspaces of a vector space V. Using Exercise 7.3, show that $X \cup Y$ is a subspace if and only if $X \subseteq Y$ or $Y \subseteq X$.

c) (Exercise 7.5) What is the smallest subspace of the space of 4×4 matrices which contains all upper triangular matrices ($a_{j,k} = 0$ for all $j > k$) and all symmetric matrices ($A = A^T$)? What is the largest subspace contained in both of those subspaces? Make sure you prove that your answers actually are subspaces if we have not done so already.

4) Let $V = c_0(\mathbb{N})$, the vector space of all complex sequences that converge to zero. Recall the map $T : V \to V$ defined by

$$T((\alpha_i)_{i=1}^{\infty}) = (0, \alpha_1, \alpha_2, \alpha_3, \dots)$$

for $(\alpha_i)_{i=1}^{\infty} \in V$.

a) Prove that T is left-invertible and exhibit the inverse.

b) Prove that T is not right-invertible. This shows that, unlike the finite-dimensional case, left-invertibility does not imply right-invertibility in general.

c) Find a linear operator $T : V \to V$ that is right invertible but not left invertible. No proof required.

5) We know from results in class that $M_2(\mathbb{R})$ and $\mathbb{R}^4$ are isomorphic as real vector spaces. Now consider the explicit isomorphism T induced by

$$e_1 \mapsto e_{1,1},\ e_2 \mapsto e_{2,1},\ e_3 \mapsto e_{1,2},\ e_4 \mapsto e_{2,2}.$$

We can define a linear map S_A from $M_2(\mathbb{R})$ to itself by taking $A, B \in M_2(\mathbb{R})$ and setting

$$S_A(B) = AB.$$

Under the isomorphism T, S_A becomes a linear map from $\mathbb{R}^4$ to $\mathbb{R}^4$, hence S_A is represented by an element of $M_4(\mathbb{R})$.

a) Find the matrix of S_A under this isomorphism, with respect to the standard basis, and check that your answer is correct.

b) Find an explicit form for all matrices in $M_4(\mathbb{R})$ that commute with the image of every S_A under T and show that the set of all such matrices is also isomorphic to $\mathbb{R}^4$ as a real vector space.

Extra Credit: I will accept no written solutions. You must explain your proof to me in my office.

A matrix $A \in M_n(\mathbb{C})$ is *nilpotent* if $\exists k \in \mathbb{N}$, $A^k = 0$, where A^k is A multiplied k times with itself. For example,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is a nilpotent matrix since $A^2 = 0$.

a) (5 points) Prove that there does not exist $A \in M_2(\mathbb{C})$ with $A^3 = 0$ but $A^2 \neq 0$.

b) (5 points) For every $k \in \mathbb{N}$, $k \leq n$, prove that there exists a nilpotent matrix $A_k \in M_n(\mathbb{C})$ such that $A_k^k = 0$ but $A_k^{k-1} \neq 0$.