Math 413/513 Assignment 4

Due Tuesday, November 4

1) (Exercise 7.6) Prove that if the product AB of two $n \times n$ matrices is invertible, then both A and B are invertible. There is a hint in the text, but you shouldn't need it.

2) Compute the rank and nullity, with supporting calculations, for the following examples, then calculate the matrix in both the standard and the indicated basis $\mathcal{B}$:

a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ 2a_1 - a_2 \end{pmatrix}$, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ a - 2c \end{pmatrix}$, $\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$

3) Let A and B be similar matrices in $M_n(\mathbb{C})$.

a) (Exercise 8.5) Prove that $\mathrm{tr}(A) = \mathrm{tr}(B)$.

b) (Exercise 3.6) Prove that $\det(A) = \det(B)$.

4) Let $A \in M_2(\mathbb{R})$.

a) Suppose that A is invertible. Compute the vectors Ae_1 and Ae_2, then show that the area of the parallelogram determined by these two vectors is equal to $|\det(A)|$. If you use a fancy theorem to compute the area, you'd better prove it.

b) Now suppose that $A \neq 0$ is not invertible. Prove that Ae_1 and Ae_2 are linearly dependent, and hence, span a line. The phrase "without loss of generality" should come in handy.

5) Let $\sigma \in S_n$. If $x = \sum_{i=1}^{n} \alpha_i e_i \in \mathbb{C}^n$, define $U_\sigma : \mathbb{C}^n \to \mathbb{C}^n$ by

$$U_\sigma(x) = \sum_{i=1}^{n} \alpha_i e_{\sigma(i)}.$$

a) (Problem 4.2, Chapter 3) Show that U_σ is always inevertible.

b) (Problem 4.3, Chapter 3) Show that for some $N > 0$, $U_\sigma^N = I_n$. Use the fact that there are only finite many permutations.

Extra Credit: I will accept no written solutions. You must explain your proof to me in my office.

Prove the analagous result for 4a) in 3 dimensions. Namely, if $A \in M_3(\mathbb{C})$ is invertible, prove that the volume of the parallelepiped spanned by Ae_1, Ae_2, and Ae_3 is equal to $|\det(A)|$. Again, if you use a formula to compute the volume, you'd better be prepared to prove your formula.