Math 413/513 Assignment 5

Due Tuesday, November 18

1) (Exercise 1.10, Chapter 4) Prove that the determinant of a matrix A is the product of its eigenvalues (counting multiplicities). **Hint:** first show that $\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda)$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues (counting multiplicities). Then compare the free terms (terms without λ) or plug in $\lambda = 0$ to get the conclusion.

2) (Exercise 2.13, Chapter 4) a) Consider the transformation $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$, $T(A) = A^t$. Find all its eigenvalues and eigenvectors. Is it possible to diagonalize this transformation? There is a **Hint** in the text.

b) Can you do the same problem, but in $M_n(\mathbb{C})$? If yes, prove it. If not, explain why not.

3) Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in M_2(\mathbb{R})$. Define a map $H_A : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$H_A(B) = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}$$

for all $B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \in M_2(\mathbb{R})$.

a) Prove that H_A is a linear map.

b) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Find the eigenvalues of H_A.

4) (Exercise 1.9, Chapter 5) Consider the space $\mathbb{R}^2$ with the norm $\|\cdot\|_p$, introduced in Section 1.5. For $p = 1, 2, \infty$, draw the "unit ball" B_p in the norm $\|\cdot\|_p$:

$$B_p := \{\mathbf{x} \in \mathbb{R}^2 \ : \ \|\mathbf{x}\|_p \le 1\}.$$

Can you guess what the balls B_p for other p look like? *Admonishment:* No laughing at this question!

5) Let $C([0,1])$ be the vector space over $\mathbb{R}$ of continuous functions from $[0,1]$ to $\mathbb{R}$. Define, for $f \in C([0,1])$,

$$\|f\|_1 = \int_0^1 |f(x)| \, dx$$

where the integral used is the ordinary Riemann integral. Show that $\| \cdot \|_1$ gives a norm on $C([0,1])$. You may make use of any properties of the integral that you learned in Calculus I.

6) Let V be a complex inner-product space.

a) Prove that if $S \subseteq V$, then $S \subseteq (S^\perp)^\perp$.

b) Prove that if V is finite-dimensional and W is a subspace of V, then $(W^\perp)^\perp = W$.

c) If $V = \ell_2(\mathbb{N})$, find a subspace W of $\ell_2(\mathbb{N})$ with W properly contained in $(W^\perp)^\perp$.

(Extra Credit) With the notation as in problem #3, let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and define

$$\|H_A\|_\infty = \max_{B \in M_2(\mathbb{R}), \|B\|=1} \|H_A(B)\|.$$

Compute $\|H_A\|_\infty$.