Math 413/513 Assignment 6

Due Thursday, December 4

1) (Exercise 3.8, Chapter 5) Let P be the orthogonal projection onto a subspace E of an inner product space V, $\dim(V) = n$, $\dim(E) = r$. Find the eigenvalues and the eigenvectors (eigenspaces). Find the algebraic and geometric multiplicity of each eigenvalue. Do all of this WITH PROOF.

2) You saw pretty much this same problem in homework #2 for $A \in M_2(\mathbb{R})$ and it was a pain to work out then. We have enough technology now to trivialize it. Let $A \in M_n(\mathbb{C})$.

a) Suppose $\exists x \in \mathbb{C}^n$, $x \neq \vec{0}$, $Ax = \vec{0}$. Must there exist $y \in \mathbb{C}^n$, $y \neq \vec{0}$, such that $A^*y = \vec{0}$? Prove or exhibit a counterexample.

b) Let $x \in \mathbb{C}^n$, $x \neq \vec{0}$. Show that $Ax = \vec{0}$ if and only if $A^*Ax = \vec{0}$.

3) a) (Exercise 6.1, Chapter 5) Orthogonally diagonalize the following matrices:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

b) (Exercise 6.4, Chapter 5) Show that a product of unitary (orthogonal) matrices is unitary (orthogonal) as well.

4) Let A and B be self-adjoint matrices in $M_n(\mathbb{C})$.

a) By considering $n = 2$, show that AB need not be self-adjoint.

b) Prove that AB is self-adjoint if and only if $AB = BA$.

5) Let u be an orthogonal matrix in $M_2(\mathbb{R})$.

a) Prove that if $\{b_1, b_2\}$ is an orthonormal basis of $\mathbb{R}^2$, then $u(b_2)$ is determined up to a negative sign by $u(b_1)$.

b) Using a), determine the matrix in the standard basis $\{e_1, e_2\}$ for every such u.

6) (Caley-Hamilton for diagonalizable matrices) Let $A \in M_n(\mathbb{C})$ be diagonalizable. Prove that if p is the characteristic polynomial of A, then $p(A) = 0$.

Extra Credit: As usual, I will accept no written work, you must present the proof to me in my office on the board. Let V be the vector space of all real-valued differentiable functions on the interval $(0,1)$ that are continuous on $[0,1]$, and whose derivative is also differentiable on $(0,1)$ and continuous on $[0,1]$. Define $D : V \to V$,

$$D(f)(x) = f'(x)$$

for all $f \in V$, $x \in (0,1)$.

a) Find all eigenvalues of D.

b) If we define a norm on V by $\|f\|_\infty = \max\limits_{x\in[0,1]} |f(x)|$, show that D is not a continuous map from $(V, \|\cdot\|_\infty)$ to itself. It may behoove you to look up "bounded linear operator."