Math 413/513 Final

Wednesday, December 18th

1) a) State the subspace test.

b) Define a linear map $T : V \to W$ where V and W are vector spaces over a field $\mathbb{F}$.

2) a) Let $A \in M_n(\mathbb{C})$ and let $\mathcal{S} = \{B \in M_n(\mathbb{C}) \mid BA = AB\}$. Prove that $\mathcal{S}$ is a subspace of $M_n(\mathbb{C})$.

b) Define $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, $T(A) = \dfrac{A - A^t}{2}$ where A^t is the transpose of A. Show that T is linear.

3) a) State the Rank/Nullity Theorem.

b) Define an eigenvalue for a linear map $T : V \to V$ where V is a finite-dimensional vector space over $\mathbb{C}$.

4) a) Prove that if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and $n > m$, then T is not one-to-one.

b) Define $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, $T(A) = \dfrac{A - A^t}{2}$ where A^t is the transpose of A. From 2b), this map is linear. Determine all eigenvalues of T, with proof.

5) Give the definition of the determinant of an $n \times n$ matrix $A = (a_{i,j})_{i,j=1}^n \in M_n(\mathbb{C})$

6) Prove that if $A = (a_{i,j})_{i,j=1}^{n} \in M_n(\mathbb{C})$ and $B = (b_{i,j})_{i,j=1}^{n} \in M_n(\mathbb{C})$ is obtained from A by interchanging two rows of A, then $\det(A) = -\det(B)$.

7) a) Define the characteristic polynomial of a matrix $A \in M_n(\mathbb{C})$.

b) Define what it means for a vector space V over a field $\mathbb{F}$ to be infinite-dimensional.

8) Do ONE of the following two problems. If you attempt both, I will grade the problem you do WORSE on.

a) Let $A \in M_n(\mathbb{C})$ be normal. Show that if $p(x) = \sum_{i=0}^{n} \alpha_i x^i$ is the characteristic polynomial of A, then $\sum_{i=0}^{n} \alpha_i A^i = 0_{M_n(\mathbb{C})}$, where $A^0 = I_{M_n(\mathbb{C})}$. BONUS: show this for any $A \in M_n(\mathbb{C})$, no assumption of normality.

-OR-

b) Let V be an infinite-dimensional vector space over a field $\mathbb{F}$. Show that $L(V) = \{T : V \to V \mid \text{T is linear}\}$ is also infinite-dimensional. No points for showing $L(V)$ is a vector space!