Math 413/513 Midterm

IN-CLASS PORTION

Tuesday, October 22nd

1) a) (5 points) Define a vector space V over a field $\mathbb{F}$.

b) (4 points) For two different fields $\mathbb{F}_1$ and $\mathbb{F}_2$, give an example of a vector space over $\mathbb{F}_1$ and a vector space over $\mathbb{F}_2$. Be sure to state which field your examples are over.

2) a) (3 points) Define a subspace W of a vector space V over a field F.

b) (4 points) State the subspace test.

c) (5 points) Provide two distinct, proper, nonzero subspaces of $\mathbb{R}^4$ over $\mathbb{R}$.

3) a) (4 points) Define what it means for a subset S of a vector space V over $\mathbb{F}$ to be linearly independent over $\mathbb{F}$.

b) (3 points) Define a basis for a vector space V over a field $\mathbb{F}$.

c) (2 points) Define the dimension of a vector space V over a field $\mathbb{F}$.

d) (4 points) Give an example of an infinite dimensional vector space over $\mathbb{R}$.

4) a) (6 points) Define a norm on a vector space V over $\mathbb{R}$ or $\mathbb{C}$.

b) (4 points) Give two norms on $\mathbb{R}^2$.

5) a) (7 points) Define an inner-product space V over $\mathbb{R}$ or $\mathbb{C}$.

b) (4 points) Define an orthogonal set S in an inner-product space V.

c) (5 points) In general terms, describe what the Gram-Schmidt Theorem says. It is sufficient, but not necessary, to state the theorem.

6) (15 points) Let V be the vector space $C(\mathbb{R})$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, considered as a vector space over $\mathbb{R}$. Let

$$W = \{f \in C(\mathbb{R}) |\ f'(x) \text{ exists } \forall\ x \in \mathbb{R}\}$$

where f' is the ordinary derivative. Show that W is a subspace of V. You may assume the rules of differentiation.

Math 413/513 Midterm

TAKE-HOME PORTION

Due Thursday, October 31

GENERAL RULES: You may use your textbook and notes as resources, but no other texts and certainly NO PEOPLE other than yourself and NO INTERNET aside from the notes.

1) Let $V = \mathbb{R}^2$ as a vector space over $\mathbb{R}$. Consider the subset S of $\mathbb{R}^2$ defined by

$$S = \{(x, y) \ : \ |x| = |y|\}$$

.

a) Show that S is not a subspace of $\mathbb{R}^2$.

b) Prove that the intersection of S with any proper subspace of $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

2) Recall the definitions of the norms $\|\cdot\|_p$ on $\mathbb{R}^n$ for $1 \leq p < \infty$. For $v \in \mathbb{R}^n$, define

$$\|v\| = \|v\|_2 + \|v\|_1.$$

a) Show that $\|\cdot\|$ is a norm on $\mathbb{R}^n$ for all $n \geq 1$.

b) Prove that $\|\cdot\|$ is not induced by an inner product on $\mathbb{R}^n$ for any $n > 1$; i.e., prove that there exists no inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ with $\sqrt{\langle v, v \rangle} = \|v\|$ for all $v \in \mathbb{R}^n$.

3) Let V be a finite-dimensional vector space of ODD dimension $n > 2$ over $\mathbb{R}$. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for V over $\mathbb{R}$. Show that the set

$$\{x_1 + x_2, x_2 + x_3, \ldots, x_{n-1} + x_n, x_n + x_1\}$$

is also a basis for V over $\mathbb{R}$.

4) Consider the vector space $\mathcal{P}$ over $\mathbb{R}$ of all polynomials with real coefficients. If $p(x), q(x) \in \mathcal{P}$, define

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x) \ dx$$

where the notation is the ordinary Riemann integral. If you don't remember how to integrate polynomials, write $p(x) = \sum_{k=0}^{n} \alpha_k x^k$ and $q(x) = \sum_{k=0}^{m} \beta_k x^k$. Then

$$\langle p(x), q(x) \rangle = \sum_{k=0}^{n+m} \frac{\gamma_k}{k+1}$$

where γ_k is the coefficient of x^k of the polynomial $p(x)q(x)$.

Let $S = \{1\} \subset \mathcal{P}$ and define the polynomials

$$t_n(x) = (n+1)x^n - (n+2)x^{n+1}$$

for $n \in \mathbb{N} \cup \{0\}$. Let

$$\mathcal{B} = \{t_n(x)\}_{n=0}^{\infty}.$$

a) Check that $\mathcal{B} \subseteq S^{\perp}$.

b) Prove that $\mathcal{B}$ is a basis for $S^{\perp}$.