Math 413/513 Final

Wednesday, December 17th

1) a) (9 points) State the subspace test.

b) (6 points) Define a linear map $T : V \to W$ where V and W are vector spaces over a field $\mathbb{F}$.

2) a) (15 points) Let $\mathcal{S} = \{A = (a_{i,j})_{i,j=1}^n \in M_n(\mathbb{C}) \mid a_{i,i} = 0 \ \forall \ 1 \le i \le n\}$. Prove that $\mathcal{S}$ is a subspace of $M_n(\mathbb{C})$.

b) (10 points) Define $T : \mathbb{C}^n \to \mathbb{C}$, $T((\alpha_i)_{i=1}^n) = \alpha_n$. Show that T is linear.

3) a) (7 points) Define an orthogonal projection $P : V \to W$ where V is a finite-dimensional inner-product space and $W \subseteq V$ is a subspace.

b) (7 points) State what it means for $A \in M_n(\mathbb{C})$ to be diagonalizable.

4) a) (18 points) Prove that the orthogonal projection onto a subspace E of a finite-dimensional inner product space V is unique, i.e., does not depend on the choice of orthonormal basis.

b) (12 points) Let $A \in M_n(\mathbb{C})$ and let $B = A + A^*$. Prove that B is diagonalizable.

5) (7 points) Define an eigenvalue for a matrix $A \in M_n(\mathbb{C})$.

6) (30 points) Let $A, B \in M_n(\mathbb{C})$ be invertible and suppose $ABA = B$. Prove that if v is an eigenvector for A, then Bv is also an eigenvector for A.

EXTRA CREDIT: Show that A and B^2 have a common eigenvector.

7) a) (5 points) Define a nilpotent matrix $A \in M_n(\mathbb{C})$.

b) (9 points) Define the adjoint of a linear map $T : V \to V$ where V is an inner product space.

8) (25 points each) Do ONE of the following two problems. If you attempt both, I will grade the problem you do WORSE on.

a) Let $A \in M_n(\mathbb{C})$ be nilpotent. Show that if $p(x) = \sum_{i=0}^{n} \alpha_i x^i$ is the characteristic polynomial of A, then $\sum_{i=0}^{n} \alpha_i A^i = 0_{M_n(\mathbb{C})}$, where $A^0 = I_{M_n(\mathbb{C})}$.

-OR-

b) Define $T : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$,

$$T((\alpha_n)_{n=1}^{\infty}) = \left(\frac{\alpha_n}{n^2}\right)_{n=1}^{\infty}.$$

Determine T^* and show that $\ker(T^*)^{\perp} \neq \operatorname{ran}(T)$.