Math 413/513 Midterm

IN-CLASS PORTION

Tuesday, October 14th

1) a) (8 points) Define a vector space V over $\mathbb{R}$ or $\mathbb{C}$.

b) (5 points) Define a subspace W of a vector space V over $\mathbb{R}$ or $\mathbb{C}$.

2) a) (5 points) Define a basis for a vector space V over $\mathbb{R}$ or $\mathbb{C}$.

b) (2 points) Define the dimension of a vector space V over $\mathbb{R}$ or $\mathbb{C}$.

c) (5 points) Give an example of an infinite dimensional vector space with a countable basis.

3) a) (6 points) Define a linear map $T : V \to W$ where V and W are vector spaces over the same field (either $\mathbb{R}$ or $\mathbb{C}$).

b) (3 points) Define an isomorphism $T : V \to W$ where V and W are vector spaces over the same field (either $\mathbb{R}$ or $\mathbb{C}$).

c) (5 points) Give two nonisomorphic, nonzero proper subpaces of $\mathbb{R}^5$

4) a) (5 points) State the Hausdorff Maximality Principle.

b) (2 points) Give examples of two results that are logically equivalent to the Hausdorff Maximality Principle.

5) a) (5 points) Define the echelon form of a matrix $A \in M_{n\times m}(\mathbb{C})$.

b) (3 points) Define a pivot for a matrix $A \in M_{n\times m}(\mathbb{C})$.

c) (6 points) Let $A \in M_{3\times 7}(\mathbb{C})$. Find the matrix $B \in M_3(\mathbb{C})$ that adds twice the second row of A to the third row, then interchanges the first and second row.

6) (15 points) Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, \text{ or } z = 0\}$. Prove that W is NOT a subspace of $\mathbb{R}^3$.

Math 413/513 Midterm

TAKE-HOME PORTION

Due Tuesday, October 21

GENERAL RULES: You may use your textbook and notes as resources, but no other texts and certainly NO PEOPLE other than yourself and NO INTERNET aside from the notes or book.

1) Let $\alpha \in \mathbb{R}$. Consider the subset $\mathcal{S}$ of $\mathbb{R}^n$ consisting of all vectors with $\sum_{i=1}^{n} x_i = \alpha$. Prove that S is a subspace of $\mathbb{R}^n$ if and only if $\alpha = 0$.

2) Let c denote the vector space of all convergent sequences over $\mathbb{C}$. Define a map $T : c \to c$ by

$$T((a_n)_{n=1}^{\infty}) = \left(\frac{a_n}{n}\right)_{n=1}^{\infty}.$$

a) Prove that T is linear.

b) Establish that T is injective.

c) Finally, show that T is not an isomorphism.

3) Let $A \in M_n(\mathbb{C})$. Prove that A has n pivots in its echelon form if and only if A^t has n pivots in its echelon form.

4) Let W be the subspace of $M_n(\mathbb{C})$ containing all matrices of the form $AB - BA$ with $A, B \in M_n(\mathbb{C})$. Prove that $\dim(W) = n^2 - 1$.