Math 413/513 Final

Wednesday, December 19th

The odd-numbered problems are definitions meant to aid you in the subsequent even-numbered problem. Use them wisely.

1) a) State the subspace test.

b) Define a linear map $T : V \to W$ where V and W are vector spaces over a field $\mathbb{F}$.

2) Let V be a vector space over a field $\mathbb{F}$ and let $T : V \to V$ be linear. Let

$$W = \{x \in V \mid T(x) = x\}.$$

Prove that W is a subspace of V.

3) a) Define an orthogonal projection $P \in M_n(\mathbb{R})$.

b) Let V be an inner product space over $\mathbb{R}$. Let x and y be vectors in V. Define what it means for x and y to be orthogonal.

4) a) Let P and Q be orthogonal projections in $M_n(\mathbb{R})$. Show that PQ is an orthogonal projection if and only if $PQ = QP$.

b) Let V be an inner product space over $\mathbb{R}$ and x and y be vectors in V. Show that if $\|x\| = \|y\|$, then $x - y$ is orthogonal to $x + y$.

5) a) Let $V_1, V_2, ..., V_n$ be vector spaces over $\mathbb{R}$. Define $V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

b) Let V be a vector space over a field $\mathbb{F}$. Define an eigenvalue for a linear operator $T : V \to V$.

6) a) Prove that $M_2(\mathbb{R}) \otimes \mathbb{R}^5$ and $\mathbb{R}^{10} \otimes \mathbb{R}^2$ are isomorphic as vector spaces over $\mathbb{R}$.

b) Let $A \in M_n(\mathbb{R})$ and suppose $\exists k \in \mathbb{N}$, $A^k = 0_n$ (here, A^k is the product of A with itself k times). Show that $\lambda = 0$ is the only eigenvalue of A.

EXTRA CREDIT: With A as in part b) above, prove that $k \leq n$.

7) a) Define a basis for a vector space V over a field $\mathbb{F}$.

b) Define the dimension of a vector space V over a field $\mathbb{F}$.

8) Do ONE of the following two problems. If you attempt both, I will grade the problem you do WORSE on.

a) Let V be a two-dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be linear. Suppose T has eigenvalues λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$. Show that there is a vector $x \in V$ such that $\{x, Tx\}$ is a basis for V.

-OR-

b) Let V be an n-dimensional vector space over a field $\mathbb{F}$ and let W_1 and W_2 be $(n-1)$-dimensional subspaces of V with $W_1 \neq W_2$. Show that $\dim(W_1 \cap W_2) = n - 2$.