Math 413/513 Midterm Part 1

Monday, October 22nd

1) a) Define a vector space V over a field $\mathbb{F}$.

2) a) Define a subspace W of a vector space V over a field F.

b) State the subspace test.

3) a) Define a basis for a vector space V over a field $\mathbb{F}$.

b) Define the dimension of a vector space V over a field $\mathbb{F}$.

c) Give an example of a four dimensional vector space over $\mathbb{R}$.

d) Give an example of an infinite dimensional vector space over $\mathbb{R}$.

4) a) For vector spaces V and W over a field $\mathbb{F}$, give the definition of a linear map $T : V \to W$.

b) Provide a characterization of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, considered as vector spaces over $\mathbb{R}$.

5) a) Define an isomorphism $T : V \to W$ where V and W are vector spaces over $\mathbb{R}$.

b) Give two isomorphic subpaces of $M_2(\mathbb{R})$.

6) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 73x - 20y - 51z \\ -20x + 208y + 204z \\ -51x + 204y + 289z \end{bmatrix}.$$

Prove that T is linear.

Math 413/513 Midterm Part 2

Wednesday, October 24th

1) Let $V = M_n(\mathbb{R})$, considered as a vector space over $\mathbb{R}$. Fix $A \in M_n(\mathbb{R})$ and let

$$W_A = \{B \in M_n(\mathbb{R}) \mid AB = BA\}.$$

Show that W_A is a subspace of $M_n(\mathbb{R})$ for ALL choices of A.

2) Let $V = \mathcal{F}(\mathbb{R})$, the functions from $\mathbb{R}$ to $\mathbb{R}$, considered as a vector space over $\mathbb{R}$. For $f, g \in \mathcal{F}(\mathbb{R})$, define $T_f : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ by

$$T_f(g) = g \circ f.$$

Prove that T_f is linear.

3) Do ONE of the following two questions. If you do both, I will grade the problem you do WORSE on.

a) Let V and W be vector spaces over $\mathbb{R}$ and suppose $T : V \to W$ is an *injective* linear map. Prove that if $\mathcal{B}$ is a linearly independent subset of V, then $T(\mathcal{B})$ is a linearly independent subset of W.

-OR-

b) Prove that if $\{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then $\{v_1, 2v_2, 3v_3, \ldots, nv_n\}$ is also a basis for V.