1 a)

$+1.5$

$(1,-1) \in S$ and

$+1.5$

$(1,1) \in S$, but

$(1,-1)+(1,1)$ $+2$

$= (2,0) \notin S$

b)

$S = W_1 \cup W_2$

where $W_1 = \{(x,y) \mid y = x\}$

and $W_2 = \{(x,y) \mid y = -x\}$.

A proper subspace of

$\mathbb{R}^2$ is one of

either

i) $\{(0,0)\}$ +1

ii) $\{(x,y) \mid y = mx\}$ for some $m \in \mathbb{R}$ +2

iii) $\{(x,y) \mid x = 0\}$ +1

In case i),

$\{(0,0)\} \cap S = \{(0,0)\}$ ✓ +1

In case iii),

$$\{(x,y) \mid x=0\} \cap S$$
$$= \{(0,0)\}. \quad +.5$$

In case 2, if $m \neq \pm 1$,

then

$$\{(x,y) \mid y=mx\} \cap S$$
$$= \{(0,0)\}. \quad +.5$$

Finally,

$$S \cap \{(x,y) \mid x=y\}$$

$$= \{(x,y) \mid x=y\}$$ +1.5

$$S \cap \{(x,y) \mid x=-y\}$$

$$= \{(x,y) \mid x=-y\}$$ +1.5

So the intersection of S with any proper subspace of $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

#2 $\|x\| = \|x\|_1 + \|x\|_2 \geq 0$

a) i) $\|0_{\mathbb{R}^n}\|$

$= \|0_{\mathbb{R}^n}\|_1 + \|0_{\mathbb{R}^n}\|_2$

$= 0 + 0 = 0$ +1

If $\|x\| = 0$, then

$\|x\|_1 + \|x\|_2 = 0$

$\Rightarrow \|x\|_1 = \|x\|_2 = 0$ +1

$\Rightarrow x = 0_{\mathbb{R}^n}$ +1

since $\|\cdot\|_1, \|\cdot\|_2$ are norms.

ii) If $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$,

$$\|\alpha x\| = \|\alpha x\|_1 + \|\alpha x\|_2$$

$$\checkmark 1$$

$$= |\alpha| \, \|x\|_1 + |\alpha| \, \|x\|_2$$

$$= |\alpha| \left(\|x\|_1 + \|x\|_2 \right)$$

$$= |\alpha| \, \|x\| \quad \checkmark 1$$

since $\| \ \|_1$, $\| \ \|_2$

are norms

iii) $x, y \in \mathbb{R}^n$

$$\|x+y\| = \|x+y\|_1 + \|x+y\|_2$$

$+ 2$

$$\leq \|x\|_1 + \|y\|_1 + \|x\|_2 + \|y\|_2$$

$$= \|x\|_1 + \|x\|_2 + \|y\|_1 + \|y\|_2$$

$$= \|x\| + \|y\| + 1$$

Since $\| \|_1$ + $\| \|_2$ are norms.

b) Consider $+2$

parallelogram

$x = e_1$, $y = e_2$.

$+2$ $+2$

Then

$$\frac{\|x+y\|^2 + \|x-y\|^2}{2}$$

$$= \frac{(\|e_1+e_2\|_1 + \|e_1+e_2\|_2)^2}{2}$$

$$+ \frac{(\|e_1-e_2\|_1 + \|e_1-e_2\|_2)^2}{2}$$

$$= (2+\sqrt{2})^2 = 6 + 4\sqrt{2}$$ $+1$

$$\|e_1\|^2 + \|e_2\|^2$$

$$= \left(\|e_1\|_1 + \|e_1\|_2\right)^2 + \left(\|e_2\|_1 + \|e_2\|_2\right)^2$$

+1

$$= 8 \neq 6 + 4\sqrt{2}$$ +1

So the parallelogram property is not satisfied. Hence, $\|\cdot\|$ is not induced by an inner product. +1

#3 Let $\{x_1, x_2, \dots, x_{2n+1}\}$ be a basis for V.

Suppose

$$\left(\sum_{i=1}^{2n} \alpha_i (x_i + x_{i+1})\right) + \alpha_{2n+1}(x_{2n+1} + x_1)$$

$$= 0_V + 2$$

Rearranging, we get

$$\sum_{i=1}^{2n} (\alpha_i + \alpha_{i+1}) x_{i+1} + (\alpha_1 + \alpha_{2n+1}) x_1 = 0_V$$

+3

$$\Rightarrow \alpha_i + \alpha_{i+1} = 0$$

+2

$\forall \; 1 \le i \le 2n$ and

$$\alpha_1 + \alpha_{2n+1} = 0.$$

Hence

$$0 = \sum_{i=1}^{2n} (-1)^i (\alpha_i + \alpha_{i+1})$$

+4

$$- (\alpha_1 + \alpha_{2n+1})$$

$$= -2\alpha_1$$

$$\Rightarrow 2\alpha_1 = 0 \Rightarrow \alpha_1 = 0$$

+2

Then $\alpha_i = 0 \ \forall$

$1 \le i \le 2n+1$

Therefore

$$\{x_1+x_2,\ x_2+x_3,\ \dots,\ x_{2n}+x_{2n+1},\ x_{2n+1}+x_1\}$$

is linearly independent.

Since $\dim(V) = n$ and this set has n elements, we get that it is a basis.

+2

#4 $t_n(x)$

$$= (n+1)x^n - (n+2)x^{n+1}$$

a)

$$\langle t_n(x), 1 \rangle$$

$$= \int_0^1 t_n(x)\,dx$$

$$= \int_0^1 \left((n+1)x^n - (n+2)x^{n+1}\right)dx$$

$$= \left(x^{n+1} - x^{n+2}\right)\Big|_0^1 = 0 \checkmark$$

+2 +2 +1

b) Suppose $p(x) \in S^{\perp}$.

Then $0 = \langle 1, p(x) \rangle$

$$= \int_0^1 p(x)\,dx. \quad +1$$

If $p(x) = \sum_{n=0}^{k} a_n x^n$,

$$+1$$

$$0 = \int_0^1 p(x)\,dx = \sum_{n=0}^{k} \frac{a_n}{n+1} \quad +1$$

$$\Rightarrow \quad a_k = -(k+1) \sum_{n=0}^{k-1} \frac{a_n}{n+1}$$

So we need

$\beta_0, \dots, \beta_{k-1}$ with

$$p(x) = \sum_{n=0}^{k-1} \beta_n t_n(x)$$

$+1$

$$= \sum_{n=0}^{k-1} \beta_n \left((n+1)x^n - (n+2)x^{n+1} \right)$$

Then the coefficient

of 1 is β_0, +1

the coefficient of

x^i, $1 \leq i < k-1$, is

+2

$(i+2)(\beta_{i+1} - \beta_i)$, and

the coefficient of

+1

x^k is $-(k+1)\beta_{k-1}$

Claim:

$$\beta_i = \sum_{n=0}^{i} \frac{a_n}{n+1} \quad (1 \le i \le k$$

+6

if $1 \le i < k$,

$$(i+2)(\beta_{i+1} - \beta_i)$$

+1

$$= (i+2)\left(\frac{a_{i+1}}{i+2}\right) = a_{i+1}.$$

If $i = k-1$,

$$-(k+1)\beta_{k-1}$$

$$= -(k+1)\sum_{n=0}^{k-1} \frac{\alpha_n}{n+1} + 1$$

$= \alpha_k + 1$ by the relationship deduced earlier.

Thus, $\{t_n(x)\}_{n=1}^{\infty}$

is spanning for $S^{\perp}$.

Now if

$$0 = \sum_{n=0}^{k} \beta_n t_n(x) \quad +1$$

$$= \sum_{n=0}^{k} \beta_n \left((n+1) x^n - (n+2) x^{n+1} \right)$$

then by prior calculation

$\beta_0 = 0$

$(n+2)(\beta_{n+1} - \beta_n) = 0$ +1

$\forall \; 1 \leq n < k$

$\Rightarrow \beta_n = 0 \;\forall\; 0 \leq n < k$

and $\quad -\beta_k (k+2) = 0$ +1

$\Rightarrow \beta_k = 0.$

Therefore,

$\{t_n(x)\}_{n=0}^{\infty}$ is

linearly independent,

and so is a basis

for $S^{\perp}$.

1) a) 5

b) 10

2) a) 8

b) 12

3) 15

4) a) 5

b) 20