Multilinear Functions & Tensors

(section 8.5)

Start: The inner product on $\mathbb{R}^n$

If $x, y, z \in \mathbb{R}^n$,

$$\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$$

$$\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$$

Also if $\alpha \in \mathbb{R}$,

$$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$$

$$= \langle x, \alpha y\rangle$$

This say that the inner product, thought of as a function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$, is linear in each variable separately.

But the inner product is not linear as a map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$: with $n=2$,

$$\left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \end{bmatrix} \right\rangle$$

$$= 16 + 0 \neq \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \end{bmatrix} + \begin{bmatrix} -9 \\ 3 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} 4 \\ 11 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \end{bmatrix} \right\rangle = 66$$

Observation: $(\mathbb{F} = \mathbb{C})$

If $n=2$, $V=\mathbb{C}^2$,

$$\left\langle \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle$$

$$= z_1 \overline{w_1} + z_2 \overline{w_2}$$

is linear in the first variable, by conjugate linear in the second. We'll only talk about vector spaces over $\mathbb{R}$ for a bit.

Definition: (cross product of vector spaces)

If $V_1, V_2, \ldots, V_n$ are vector spaces over $\mathbb{R}$, let

$$V = V_1 \times V_2 \times \cdots \times V_n$$

and define addition and scalar multiplication on V by

if $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in V$ and $\alpha \in \mathbb{R}$,

$$\boxed{\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)}$$

$$\boxed{\begin{aligned}&(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) \\ &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\end{aligned}}$$

and zero vector

$$(0_{V_1}, 0_{V_2}, \ldots, 0_{V_n}),$$

then V is a vector space over $\mathbb{R}$ under these operations.

Definition: (bilinearity)

Let V_1, V_2, V_3 be vector spaces over $\mathbb{R}$. Then a function

$$B : V_1 \times V_2 \to V_3 \quad \text{is}$$

said to be **bilinear** if

$\forall\, \alpha \in \mathbb{R},\ x_1, y_1 \in V_1,\ x_2, y_2 \in V_2,$

1) $$B((\alpha x_1, x_2)) = \alpha B(x_1, x_2) = B((x_1, \alpha x_2))$$

2) $$B((x_1 + y_1, x_2)) = B((x_1, x_2)) + B((y_1, x_2))$$

3) $B((x_1, x_2+y_2))$

$= B((x_1,x_2)) + B((x_1,y_2))$

Example: $B =$ inner product

Definition: (multilinearity)

If $V_1, V_2, \ldots, V_n, V$ are vector spaces over $\mathbb{R}$, then a function

$$M : V_1 \times V_2 \times \cdots \times V_n \to V$$

is said to be **multilinear** if it is linear in each variable.