Announcements

1) HW 6 up, due Monday

2) Final : Wednesday 12/19 11:30-2:30

Definition: (multilinearity)

If $V_1, V_2, \ldots, V_n, V$ are vector spaces over $\mathbb{R}$, then a function

$$M : V_1 \times V_2 \times \cdots \times V_n \to V$$

is said to be **multilinear** if for all $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in V_1 \times V_2 \times \cdots \times V_n$

and $\alpha \in \mathbb{R}$,

1) $\alpha M((x_1, x_2, \ldots, x_n))$

$$= M((\alpha x_1, x_2, \ldots, x_n))$$

$$= M((x_1, \alpha x_2, \ldots, x_n))$$

$$\vdots$$

$$= M((x_1, x_2, \ldots, \alpha x_n))$$

2) $M((x_1, x_2, \ldots, x_n) + (x_1, x_2, \ldots, x_{i-1}, y_i, \ldots, x_n))$

$$= M((x_1, x_2, \ldots, x_{i-1}, x_i + y_i, \ldots, x_n))$$

$$\forall\ 1 \leq i \leq n$$

Example 1: (trilinear map)

$$M: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Let $f_1: \mathbb{R} \to \mathbb{R},\ f_1(x) = 7x$

$f_2: \mathbb{R} \to \mathbb{R},\ f_2(x) = 3x$

$f_3: \mathbb{R} \to \mathbb{R},\ f_3(x) = 12x$

$$M((x_1, x_2, x_3)) = f_1(x_1) f_2(x_2) f_3(x_3)$$

$$= 252\, x_1 x_2 x_3$$

Let $\alpha \in \mathbb{R}$.

$\alpha M((x_1, x_2, x_3))$

$= \alpha \cdot 252 \cdot x_1 x_2 x_3$

$= 252 \cdot (\alpha x_1) x_2 x_3 = M((\alpha x_1, x_2, x_3))$

$= 252\, x_1 (\alpha x_2) x_3 = M((x_1, \alpha x_2, x_3))$

$= 252\, x_1 x_2 (\alpha x_3) = M((x_1, x_2, \alpha x_3))$

This shows property 1)

Now let $y \in \mathbb{R}$.

Then

$$M((x_1+y, x_2, x_3))$$

$$= 252\,(x_1+y)\cdot x_2 x_3$$

$$= 252\,(x_1 x_2 x_3 + y x_2 x_3)$$

$$= 252\,x_1 x_2 x_3 + 252\,y x_2 x_3$$

$$= M((x_1, x_2, x_3)) + M((y, x_2, x_3))$$

Similarly, we can show

$$M((x_1, x_2+y, x_3)) = M((x_1, x_2, x_3)) + M((x_1, y, x_3))$$

and

$$M((x_1, x_2, x_3+y)) = M((x_1, x_2, x_3)) + M((x_1, x_2, y))$$

and so M is trilinear.

Observation: (vector space structure)

Let $\mathcal{L}(V_1, V_2, \ldots, V_n; V)$ denote the set of all multilinear maps from $V_1 \times V_2 \times \cdots \times V_n$ to V.

If $f, g \in \mathcal{L}(V_1, V_2, \ldots, V_n; V)$ and $\alpha \in \mathbb{R}$, we can endow $\mathcal{L}(V_1, V_2, \ldots, V_n; V)$ with a vector space structure as follows:

If $(y_1, y_2, \ldots, y_n) \in V_1 \times V_2 \times \cdots \times V_n$,

$$(f+g)((y_1, y_2, \ldots, y_n))$$

$$= f((y_1, y_2, \ldots, y_n))$$
$$+ g((y_1, y_2, \ldots, y_n))$$

$$(\alpha f)((y_1, y_2, \ldots, y_n))$$

$$= \alpha \cdot f((y_1, y_2, \ldots, y_n))$$

Zero vector is the zero function

$$0((y_1, y_2, \ldots, y_n)) = 0_V$$

Definition: (tensor & tensor product)

Given vector spaces $V_1, V_2, \ldots, V_n$ over the tensor product of $V_1, V_2, \ldots, V_n$ is the vector space

$$\mathcal{L}(V_1^*, V_2^*, \ldots, V_n^*; \mathbb{R})$$

and is denoted by

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

A tensor is merely an element of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$

Note: This construction is sometimes referred to as the algebraic tensor product. If one of the V_i's is infinite-dimensional, we usually impose additional restrictions on the tensor.

Example 2:

$$\mathbb{R} \otimes \mathbb{R} = \mathcal{L}(\mathbb{R}^*, \mathbb{R}^*; \mathbb{R})$$

$$= \mathcal{L}(\mathbb{R}, \mathbb{R}; \mathbb{R})$$

What is a bilinear function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$?

Let $(x, y) \in \mathbb{R} \times \mathbb{R}$

If $B: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is bilinear,

$$
\begin{aligned}
& B((x,y)) \\
&= B((x,y)) \\
&= x\, B((1,y)) \\
&= xy\, B((1,1))
\end{aligned}
$$

$\Rightarrow$ $B((x,y))$ is completely determined by $B((1,1))$ and the values of x & y.

We can write

$$B((x,y)) = xy\,B((1,1))$$
$$= \langle x,y \rangle\, B((1,1))$$

$\Rightarrow$ B is a scalar multiple of the inner product.

$\Rightarrow$ $\dim_{\mathbb{R}}(\mathbb{R}\otimes\mathbb{R}) = 1,$

and so $\mathbb{R}\otimes\mathbb{R}$ is isomorphic to $\mathbb{R}$ as a vector space!

In general, if $M : \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \to \mathbb{R}$

is multilinear,

$$M((x_1, x_2, \ldots, x_n))$$
$$= x_1 x_2 \cdots x_n \cdot M((1, 1, \ldots, 1))$$

and so all $M \in \underbrace{\mathbb{R} \otimes \cdots \otimes \mathbb{R}}_{n \text{ times}}$

are scalar multiples of

$$f((x_1, x_2, \ldots, x_n)) = x_1 x_2 \cdots x_n,$$

and so $\dim_{\mathbb{R}}(\mathbb{R} \otimes \mathbb{R} \otimes \cdots \otimes \mathbb{R}) = 1$!

Example 3: Now let's consider

$$\mathbb{R}^2 \otimes \mathbb{R}^2 = \mathcal{L}((\mathbb{R}^2)^*, (\mathbb{R}^2)^*; \mathbb{R})$$

$$= \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$$

Let $f \in \mathbb{R}^2 \otimes \mathbb{R}^2$

$$f(((x_1, y_1), (x_2, y_2)))$$

$$= f(((x_1, 0) + (0, y_1)), (x_2, y_2)))$$

$$= f(((x_1, 0), (x_2, y_2)))$$
$$+ f(((0, y_1), (x_2, y_2)))$$

(additivity in 1st coordinate)

$$= f(((x_1, 0), (x_2, 0) + (0, y_2)))$$

$$+ f(((0, y_1), (x_2, 0) + (0, y_2)))$$

$$= f(((x_1, 0), (x_2, 0)))$$

$$+ f(((x_1, 0), (0, y_2)))$$

$$+ f(((0, y_1), (x_2, 0)))$$

$$+ f(((0, y_1), (0, y_2)))$$

additivity in 2nd coordinate

$$= x_1 x_2 f(((1,0),(1,0)))$$
$$+ x_1 y_2 f(((1,0),(0,1)))$$
$$+ y_1 x_2 f(((0,1),(1,0)))$$
$$+ y_1 y_2 f(((0,1),(0,1)))$$

pull out scalars in each coordinate

So f is a linear combination of

$$g_{1,1}(((x_1,y_1),(x_2,y_2))) = x_1 x_2$$
$$g_{1,2}(((x_1,y_1),(x_2,y_2))) = x_1 y_2$$
$$g_{2,1}(((x_1,y_1),(x_2,y_2))) = y_1 x_2$$
$$g_{2,2}(((x_1,y_1),(x_2,y_2))) = y_1 y_2$$

Claim: $\{g_{i,j} \mid 1 \leq i,j \leq 2\}$

is a basis for $\mathbb{R}^2 \otimes \mathbb{R}^2$

We already know the set spans $\mathbb{R}^2 \otimes \mathbb{R}^2$, so let's prove linear independence:

Suppose $\exists\, \alpha_{i,j} \in \mathbb{R}$, $1 \leq i,j \leq 2$, with

$$\sum_{i,j=1}^{2} \alpha_{i,j}\, g_{i,j} = 0$$

Let $e_1 = (1,0)$, $e_2 = (0,1)$ be the standard basis of $\mathbb{R}^2$.

By construction,

$$g_{i,j}((e_s, e_t)) = \begin{cases} 1, & s=i, t=j \\ 0, & \text{otherwise} \end{cases},$$

so for example,

$$0 = \sum_{i,j=1}^{2} \alpha_{i,j}\, g_{i,j}((e_1, e_2))$$

$$= \alpha_{1,2}\, g_{1,2}((e_1, e_2))$$

$= \alpha_{1,2}$. In this manner, we can show $\alpha_{i,j} = 0 \ \forall\ 1 \le i,j \le 2$

$\Rightarrow \{g_{i,j} \mid 1 \le i,j \le 2\}$ is a basis.