**Definition.** (tensor of linear maps)

Let $V_1, V_2, \ldots, V_n, W_1, W_2, \ldots, W_n$ be vector spaces over $\mathbb{R}$.

Let $T_i : V_i \to W_i$, $1 \le i \le n$, be linear. Then we may define $T = T_1 \otimes T_2 \otimes \cdots \otimes T_n$,

$$T : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to W_1 \otimes W_2 \otimes \cdots \otimes W_n$$

by $(x_i \in V_i)$

$$\boxed{T(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = T_1(x_1) \otimes T_2(x_2) \otimes \cdots \otimes T_n(x_n)}$$

and extend by linearity to

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n.$$

Then $T$ is a linear map.

**Theorem**: (linear $\Leftrightarrow$ multilinear)

Let $M \in V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

Then $\exists$ a unique linear map $T : V_1^* \otimes V_2^* \otimes \cdots \otimes V_n^* \to \mathbb{R}$

with

$$\boxed{M((f_1, f_2, \ldots, f_n)) = T(f_1 \otimes f_2 \otimes \cdots \otimes f_n)}$$

$\forall f_i \in V_i^*, 1 \leq i \leq n$

proof: We will assume the $V_i$'s are finite dimensional, though the reason for this is purely notational.

Let $\{e_{i,j}\}_{j=1}^{\dim(V_i)}$ be a basis for $V_i$, $1 \le i \le n$, and let $e_{i,j}^*$, $1 \le i \le n$, $1 \le j \le \dim(V_i)$, be the dual basis:

$$e_{i,j}^*(e_{i,k}) = \delta_{j,k} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

Define

$$T(e^*_{1,j_1} \otimes e^*_{2,j_2} \otimes \cdots \otimes e^*_{n,j_n})$$

$$= M((e^*_{1,j_1}, e^*_{2,j_2}, \ldots, e^*_{n,j_n})),$$

extend to $V_1^* \otimes V_2^* \otimes \cdots \otimes V_n^*$ by linearity. Check the condition holds!

$\square$

## Example 1: (interpreting tensors)

Know: $\mathbb{R}^2 \otimes \mathbb{R}^2$ is isomorphic to $\mathbb{R}^4$ as a vector space.

How do we interpret

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

as an element of $\mathbb{R}^4$?

As follows:

$$\begin{bmatrix} 1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

Note: $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 3 \\ 6 \\ 4 \\ 8 \end{bmatrix}$$

$$\neq \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

In general, if $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$,

$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m$, then

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}_{\in \mathbb{R}^n \otimes \mathbb{R}^m} = \underbrace{\begin{bmatrix} x_1 \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \\ x_2 \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \\ \vdots \\ x_n \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \end{bmatrix}}_{\in \mathbb{R}^{nm}}$$

**Note:** this identification erases multilinearity.

Similarly, if $A, B \in M_2(\mathbb{R})$,

$A = (A_{i,j})_{i,j=1}^{2}$, $B = (B_{i,j})_{i,j=1}^{2}$,

define $A \otimes B$ as before on $\mathbb{R}^2 \otimes \mathbb{R}^2$.

As a linear map from $\mathbb{R}^4$ to $\mathbb{R}^4$,

$$A \otimes B = \begin{bmatrix} A_{1,1} B & A_{1,2} B \\ A_{2,1} B & A_{2,2} B \end{bmatrix} \in M_4(\mathbb{R})$$

i.e. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 3 & 0 \\ 9 & -8 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 0 & 6 & 0 \\ 9 & -8 & 18 & -16 \\ 9 & 0 & 12 & 0 \\ 27 & -24 & 36 & -32 \end{bmatrix}$$

$$= \begin{bmatrix} 1\begin{bmatrix} 3 & 0 \\ 9 & -8 \end{bmatrix} & 2\begin{bmatrix} 3 & 0 \\ 9 & -8 \end{bmatrix} \\ 3\begin{bmatrix} 3 & 0 \\ 9 & -8 \end{bmatrix} & 4\begin{bmatrix} 3 & 0 \\ 9 & -8 \end{bmatrix} \end{bmatrix}$$