Name:

Math 473/573 Final

1) a) (3 points) State what it means for $Q \in \mathbb{C}^{m \times m}$ to be unitary.

b) (2 points) State what it means for $P \in \mathbb{C}^{m \times m}$ to be a projection.

c) (2 points) State what it means for a projection $P \in \mathbb{C}^{m \times m}$ to be orthogonal (equivalent conditions are acceptable).

d) (5 points) State what it means for $A \in \mathbb{C}^{m \times m}$ to be Hessenberg.

2) Let $A \in \mathbb{C}^{9\times 4}$. Let $A = \hat{U}\hat{\Sigma}V^*$ and $A = U\Sigma V^*$ be the reduced and full singular value decompositions of A, respectively.

a) (6 points) What are the row and column dimensions of U,Σ, and V, respectively?

b) (6 points) Define A^+, the pseudoinverse of A, and record its row and column dimensions.

c) (6 points) Let $\hat{\Sigma} = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$. Find $\|A\|_2$, $\|A^+\|_2$, and $\|A\|_{\mathcal{F}}$.

3) Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^3$ and $b = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \in \mathbb{C}^2$. Note that b is not in the range of A.

a) (6 points) Referring to the definition of the pseudoinverse A^+ from the previous problem, compute A^+.

b) (4 points) Find the vector $x \in \mathbb{C}^2$ that minimizes $\|Ax - b\|_2$ and compute $y = Ax$.

c) (6 points) Compute $\|Ax - b\|_2$ and verify that $Ax - b$ and Ax are orthogonal.

d) (4 points) Compute P, the orthogonal projection from $\mathbb{C}^3$ onto the range of A, and verify that $y = Pb$.

4) a) (6 points) Define what it means for a problem $f : X \to Y$ to be well-conditioned.

b) (2 points) Give an example of a well-conditioned problem.

c) (3 points) Define the condition number of $A \in \mathbb{C}^{m\times m}$.

d) (7 points) For $A = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$, compute the condition number of A using $\|\cdot\|_\infty$.

5) a) (6 points) State at least one part of the fundamental axiom of floating point arithmetic.

b) (6 points) The quantity $\varepsilon_{machine}$ is defined as the smallest number such that two properties hold, one of which is the fundamental axiom of floating point arithmetic. State the other property.

c) (4 points) If $\varepsilon_{machine} = 2^{-3}$, find the closest floating point approximation to $\sqrt{2} \approx 1.41421356$.

6) a) (7 points) Define what it means for an algorithm $\tilde{f} : X \to Y$ for a problem $f : X \to Y$ to be backwards stable.

b) (4 points) Give two examples of backwards stable algorithms.

c) (2 points) Give an example of an unstable algorithm.

7) Here is an algorithm in Matlab code, $A \in \mathbb{C}^{m\times n}$.

```
for j=1:n
  v=A(:,j);
  for i=1:j-1
    r(i,j)=q(:,i)'*A(:,j);
    v=v-r(i,j)*q(:,i);
  end
  r(j,j)=norm(v,2);
  q(:,j)=v/r(j,j);
end
```

a) (4 points) What does this algorithm compute? Be specific!

b) (13 points) Determine the flop count of the algorithm.

8) a) (4 points) Explain the difference between Gaussian elimination and Gaussian elimination with partial pivoting.

b) (7 points) Define what it means for $A \in \mathbb{C}^{m \times m}$ to be diagonalizable. Is every $A \in \mathbb{C}^{m \times m}$ diagonalizable?

c) (7 points) Define the Schur Factorization of $A \in \mathbb{C}^{m \times m}$. Does every $A \in \mathbb{C}^{m \times m}$ have a Schur factorization?

9) Let $A \in \mathbb{R}^{n\times n}$ with $A = A^t$.

a) (4 points) For $x \in \mathbb{R}^n$, $x \neq 0_n$, define the Rayleigh Quotient $r(x)$ of x with respect to A.

b) (2 points) If $x \in \mathbb{R}^n$ is an eigenvector of A corresponding to the eigenvalue λ, what does $r(x)$ evaluate to?

c) (6 points) If $A = \begin{bmatrix} 2 & 5 \\ 5 & 4 \end{bmatrix}$ and $x = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, compute $r(x)$.

d) (Extra Credit, 10 points) Show that $\|A\|_2 = \max\limits_{x\in\mathbb{R}^n} |r(x)|$.

10) (16 points) We have used many times the fact that if x, y, x_0, y_0 are real numbers and

$$\frac{|x - x_0|}{|x|} = O(\varepsilon), \quad \frac{|y - y_0|}{|y|} = O(\varepsilon),$$

then

$$\frac{\|(x, y) - (x_0, y_0)\|_\infty}{\|(x, y)\|_\infty} = O(\varepsilon).$$

Prove this fact.