Math 473/573 Midterm

Due Tuesday, March 11

GENERAL RULES: You may use your textbook and notes as resources, but no other texts and certainly NO PEOPLE other than yourself and NO INTERNET aside from what is on Canvas. You may use Matlab, Mathematica, or your favorite computational resource where indicated.

1) Since everyone should do this once in their lives, let $A = \begin{bmatrix} 3 & -i \\ -2 & 5 \end{bmatrix}$.

a) Compute A^*A exactly, using any method at your disposal.

b) By hand, find the EXACT eigenvalues of A^*A, no decimals. The square roots of these eigenvalues are the singular values of A.

c) Find two orthonormal eigenvectors of A^*A BY HAND. Again, your answers should be exact.

d) Recalling that $|A| = \sqrt{A^*A}$, unitarily diagonalize $|A|$ by using b) and c).

e) Find a unitary W with $WA = |A|$ (see problem 7 on Assignment 2).

f) Finally, find the singular value decomposition of A EXACTLY. Compare your answer to Matlab's svd command.

2) a) Write a program that computes the full QR decomposition of $A \in \mathbb{C}^{m\times n}$ via Householder Triangulation (see algorithm 10.1).

b) Pick b to be the 3-vector whose first coordinate is the (gregorian) year of your birth, second coordinate is the (gregorian) month of your birth, and whose third coordinate is the day of your birth in that month. If you don't know what these are, make some numbers up. Let

$$A = \begin{bmatrix} -1 & 1 \\ 9 & 6 \\ -11 & 0 \end{bmatrix}.$$

By decomposing $A = QR$ via the algorithm obtained in a), obtain a solution for x accurate to 16 decimal places for the equation $Ax = b$ by solving $Rx = Q^*b$. Compare this to Matlab's $A\backslash b$ command

c) For all m and n, determine the exact number of flops used in algorithm 10.1. See Example 1 on the 2/6 notes if this is unclear.

3) a) Find the eigenvalues of the following matrices, accurate to 4 decimal places, using your favorite method.

i) $A = \begin{bmatrix} -1 & 16 \\ -24 & 3 \end{bmatrix}$

ii) $B = \begin{bmatrix} 2 & -1 & 4 \\ 6 & 8 & 0 \\ 9 & 22 & 6 \end{bmatrix}$

b) Let $n \in \mathbb{N}$ and $\alpha_i \in \mathbb{C}$ for $0 \leq i \leq n$. Consider the polynomials

$$r(z) = \sum_{i=0}^{n} \alpha_i z^i, \quad q(z) = \sum_{i=0}^{n} \overline{\alpha_i} z^i.$$

Let $p(z) = r(z) + q(z)$. Prove that if $w \in \mathbb{C}$ satisfies $p(w) = 0$, then $p(\overline{w}) = 0$ as well.

c) Use the result from b) to show that if $A \in M_n(\mathbb{R})$, then λ is an eigenvalue of A if and only if $\overline{\lambda}$ is an eigenvalue of A.

Name:

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1) a) (3 points) State what it means for $Q \in \mathbb{C}^{m \times m}$ to be unitary.

b) (4 points) State what it means for $P \in \mathbb{C}^{m \times m}$ to be an orthogonal projection.

c) (4 points) Define a Householder reflection $Q \in \mathbb{C}^{m \times m}$.

d) (5 points) Give the normal equations for a linear system $Ax = b$ with $A \in \mathbb{C}^{m \times n}$, $x \in \mathbb{C}^n$, and $b \in \mathbb{C}^m$.

2) Let $B \in \mathbb{C}^{5\times 2}$. Let $B = \hat{Q}\hat{R}$ and $B = QR$ be the reduced and full QR factorizations of B, respectively.

a) (4 points) What are the row and column dimensions of Q and R, respectively?

b) (4 points) What are the row and column dimensions of $\hat{Q}$ and $\hat{R}$, respectively?

c) (2 points) Is Q always invertible? Why or why not?

d) (5 points) Is $\hat{R}$ always invertible? Why or why not?

3) Let $v = \begin{bmatrix} 4 \\ 5 - i \end{bmatrix} \in \mathbb{C}^2$.

a) (6 points) Calculate $\|v\|_1$, $\|v\|_2$, and $\|v\|_\infty$.

b) (6 points) Compute the orthogonal projection $P : \mathbb{C}^2 \to \text{span}\{v\}$.

c) (5 points) Let $w = \begin{bmatrix} i \\ -8 \end{bmatrix}$. Find the closest vector in $\text{span}\{v\}$ to w.

4) Let $A = \begin{bmatrix} -4 & 1 \\ 3 & 2+2i \end{bmatrix}$. In the singular value decomposition $A = U\Sigma V^*$ of A, we have

$$\Sigma = \begin{bmatrix} \sqrt{17+2\sqrt{26}} & 0 \\ 0 & \sqrt{17-2\sqrt{26}} \end{bmatrix}.$$

a) (5 points) Find the 1-norm of A.

b) (5 points) Find the Frobenius norm of A.

c) (4 points) Find the 2-norm of A.

d) (6 points) Find the 2-norm of A^{-1}.

5) (12 points) If $Q \in \mathbb{C}^{m \times m}$ is a unitary matrix, prove that Q^t, the transpose of Q, is also a unitary matrix.